Weak Duality Theorem

- The domain of $q$ is

$$D_q = \{ \mu \mid q(\mu) > -\infty \}.$$

- Proposition: The domain $D_q$ is a convex set and $q$ is concave over $D_q$.

- Proposition: (Weak Duality Theorem) We have

$$q^* \leq f^*.$$
Duality Gap

- Duality gap = $f^* - q^*$
- If $q^* = f^*$, we say there is no duality gap
- If $q^* < f^*$, we say there is a duality gap
- If there exists a geometric multiplier, then there is no duality gap since $q(\mu^*) = f^*$
- But if there no duality gap, there may exists no geometric multiplier in case $q(\mu)$ is maximized at $\mu \rightarrow \infty$
Example I

\[ \min \ f(x) = x_1 - x_2 \]

s.t. \( g(x) = x_1 + x_2 - 1 \leq 0 \)

\[ x \in X = \{(x_1, x_2) \mid x_1 \geq 0, x_2 \geq 0\} \]

Let \( q(\mu) = \min_{x_1 \geq 0, x_2 \geq 0} \{x_1 - x_2 + \mu(x_1 + x_2 - 1)\} \)

Plot \( q(\mu) \) versus \( \mu \)

Is there a duality gap?
Example II

$$\min \ f(x) = \frac{1}{2} \ (x_1^2 + x_2^2)$$

s.t. \ \ g(x) = x_1 - 1 \leq 0

$$x \in X = \mathbb{R}^2$$

Plot $q(\mu)$ versus $\mu$

Is there a duality gap?
Example III

\[
\min \ f(x) = |x_1| + x_2 \\
\text{s.t. } g(x) = x_1 \leq 0 \\
x \in X = \{(x_1, x_2) \mid x_2 \geq 0\}
\]

Plot \( q(\mu) \) versus \( \mu \)

Is there a duality gap?
Example IV

\[
\begin{align*}
\min & \quad f(x) = x \\
\text{s.t.} & \quad g(x) = x^2 \leq 0 \\
& \quad x \in X = \mathbb{R}
\end{align*}
\]

Plot \( q(\mu) \) versus \( \mu \)
Is there a duality gap?
Example V

\[
\begin{align*}
\min & \quad f(x) = -x \\
\text{s.t.} & \quad g(x) = x - 1/2 \leq 0 \\
& \quad x \in X = \{0, 1\}
\end{align*}
\]

Plot \( q(\mu) \) versus \( \mu \)

Is there a duality gap?
Strong Duality Theorem

- Assume that $X$ is convex and the functions $f : \mathbb{R}^n \rightarrow \mathbb{R}$, $g_j : \mathbb{R}^n \rightarrow \mathbb{R}$ are convex over $X$. Furthermore, the optimal value $f^*$ is finite and there exists a vector $\bar{x} \in X$ such that

$$g_j(\bar{x}) < 0, \quad \forall \ j = 1, \ldots, r.$$

- Strong Duality Theorem: There exists at least one geometric multiplier and there is no duality gap.

- Special case: linear constraints, convex cost
General Dual-Method Framework

• Outer loop: find $\mu^*, \lambda^*$ to maximize $q(\mu, \lambda)$
  – $q(\mu, \lambda)$ may not have closed form
  – $q(\mu, \lambda)$ may not be differentiable
  – May be done efficiently in some cases
    • $q(\mu, \lambda)$ is concave over a convex set

• Inner loop: find $x$ to minimize $L(x, \mu, \lambda)$

• Even if the search is successful, it may fail to solve the original problem due to the duality gap

• Often we apply duality case by case using a combination of analysis and computation
Separable Programming

- Separable problems:

\[
\begin{align*}
\text{minimize} & \quad \sum_{i=1}^{m} f_i(x_i) \\
\text{subject to} & \quad \sum_{i=1}^{m} g_{ij}(x_i) \leq 0, \quad j = 1, \ldots, r, \\
x_i & \in X_i, \quad i = 1, \ldots, m.
\end{align*}
\]
Dual Separable Programming

maximize \( q(\mu) \)

subject to \( \mu \geq 0 \),

\[
q(\mu) = \inf_{x_i \in X_i, i=1..m} \left\{ \sum_{i-1}^{m} \left( f_i(x_i) + \sum_{i-1}^{r} \mu_j g_{ij}(x_i) \right) \right\} = \sum_{i-1}^{m} q_i(\mu)
\]

\[
q_i(\mu) = \inf_{x_i \in X_i} \left\{ f_i(x_i) + \sum_{j=1}^{r} \mu_j g_{ij}(x_i) \right\}, \quad i = 1, \ldots, m.
\]

The inner loop of finding \( q(\mu) \) is decomposed into \( m \) much smaller and independent subproblems.
Remarks on Separable Programming

• For large-scale constrained optimization, decomposition is almost a must
• Based on duality, separable programming exploits the problem structure
  – Significant reduction of computing time
  – Support parallel computation
• Limitations: duality gap, separability