Region-based Line Field Design Using Harmonic Functions

Chih-Yuan Yao, Ming-Te Chi, Tong-Yee Lee, *Senior Member, IEEE,* and Tao Ju

APPENDIX A
CRITICAL POINTS OF CONTINUOUS HARMONIC FUNCTIONS

Here we show that the harmonic functions we used to define the elementary fields have well-defined critical points. In the continuous setting, we consider a point critical if the derivative of the function there is either zero or undefined. We denote the region by R, its boundary curve by B, and the harmonic functions defined in Section 4.1 by w. The results on the critical points in the interior of R have been mostly provided by [1]. Our main effort here is to extend the results onto the boundary B.

Proposition 1: The function $w(x)$ has critical points only at the centers, the terminals, or the corners on B.

Proof: First of all, G^0 corners on B do not have a well-defined tangent direction, and hence the derivative of w is undefined there. We next study the remaining critical points for different types of elementary fields:

- *Circular Type:* Since R is simply connected, the Green's function $G(c,x)$ contains no critical point in R except at the *pole* c [1]. To show that G is also free of critical points on B (away from the corners), let us suppose on the contrary that there is some $z \in B$ with vanishing partial derivatives of G. Since B is a zero-level curve of G, there will at least one other zero-level curve coming out of z into R. Since R is free of critical points, this zero-level curve will end somewhere again on B, dividing R into two sub-regions. Consider the sub-region R' that does not contain c. R' is bounded entirely by a zero-level curve of G. However, as G is harmonic, G would be constant zero inside R', contradicting the fact that R' contains no interior critical points.
- *Elliptic and Flow Types:* When R is simply connected, Walsh (see Theorem 1 and Corollary in §9.8.1 in [1]) showed that a non-constant harmonic function $w(x)$ contains no critical point interior to R if the following holds: there exists no value s such that there are at least 4 locations on B where $w(x) = s$, and that the successive alternate arcs of B separated by these locations contain each some point of B at which $w(x) > s$ and $w(x) < s$. It is easy to see that the boundary values in our harmonic functions for Elliptic and Flow Types satisfy the above condition, and hence these functions are all free of critical points interior to R.
We use an argument similar to that in the Circular Type to show that B contains no singularities except the terminals in Elliptic and Flow Types. Note that, except the singularities, B consists of two kinds of segments: non-flow-out segments, where w is constant, and flow-out segments. Function w is differentiable are both kind of segments, including where they meet. Suppose, on the contrary, that there is some point z with vanishing partial derivatives of w in a non-flow-out segment where w assumes 0 (or equivalently, 1). There is at least one other level curve of w with value 0 coming out of z, which will end somewhere on B, since the interior of R has no critical points. By our definition, all points on B where w assumes 0 lie on one non-flow-out segment in both Elliptic and Flow Types. Hence that level curve will end on the same non-flow-out segment as z, forming a closed region bounded entirely by a level curve of w, and contradicting the lack of critical points in R. Similarly, suppose z lies on a flow-out segment where $w(z)$ assumes some value $s \in (0,1)$. There are at least two level curves of w with value s coming out of z, both ending somewhere on B. By our definition of the Flow Types, there are exactly two points on B with value s, including z (the other could be a terminal, which is a jump from 0 to 1). Hence the two level curves coming out of z would either both end at a common point, or at least one of them would end at z. In either case, a closed region bounded entirely by a level curve of w would be formed, reaching a contradiction.

□

APPENDIX B
CRITICAL POINTS OF DISCRETE HARMONIC FUNCTIONS

Here we show that the discrete harmonic functions given our boundary value conditions in Section 5 have well-defined critical points like their continuous counterparts. In the discrete setting, we consider a point critical in a piece-wise linear 2D function if the level set is not a 1-manifold there, or if the topology of the level set changes at the point. Similar to the continuous case, we will separately consider the critical points in the interior of region R and on its boundary B.

B.1 Interior

We first show a general property of any discrete harmonic function defined by Equation 4 with positive weights (e.g., the mean value weights). We call a vertex *constrained* if it is given an initial value (called a *constraint*), and *free* otherwise.

Lemma 1: Let f be the piece-wise linear function defined by the solution of Equation 4 where weights b_{ij} are all positive. Then, at any free vertex v whose 1-ring vertices do not all have the same value as v in f, there exists:

- An "uphill" path of edges connecting v to some constrained vertex, so that the values at successive vertices on the path are strictly increasing, and
- A "downhill" path of edges connecting v to some constrained vertex, so that the values at successive vertices on the path are strictly decreasing.

Proof: The positivity of b_{ij} implies that the value of v lies in the range of values of its neighbors. Since the neighbors of v don't all have the same value, then there must exist some neighbor vertices $\uparrow(v), \downarrow(v)$ such that $f(\uparrow(v)) > f(v)$ and $f(\downarrow(v)) < f(v)$. Repeating the same argument, we can find a path of edge-connected vertices $v, \uparrow(v), \uparrow(\uparrow(v)), \ldots$ with strictly increasing values until a constrained vertex is reached. Similarly, a path $v, \downarrow(v), \downarrow(\downarrow(v)), \ldots$ with strictly decreasing values exists and ends at a constrained vertex. □

An immediate implication of Lemma 1 is that, for any free vertex v with a "non-flat" neighborhood, there exists some constraints that are above $f(v)$ and some below $f(v)$. We will use this observation frequently in the proof.

The specific way that we assign the constraints is the key that leads to the lack of critical points in our discrete harmonic functions. In particular, there are two important observations of our constraints that we utilize in the proof:

Lemma 2: For any type of elementary field,

1) If two neighboring constrained vertices on B share a common constraint value, that value is either the maximum or minimum among all constraints in the field.
2) There does not exist four distinct vertices v_1, v_2, v_3, v_4 on B, in counter-clockwise order, such that the constraints at both v_1, v_3 are strictly greater than those at v_2, v_4.

Now we present the main result. Let w be the discrete harmonic function computed in any of our elementary fields, we have:

Proposition 2: In the interior of R, the function $w(x)$ has no critical points other than the center c in Cyclic Type.

Proof: We will show, in order, that critical points do not exist inside any triangle, along any interior edge, or at any interior vertex except the center.

In a triangle: A critical point exists inside a triangle only if all vertices of the triangle have the same value (i.e., "flat"). Otherwise the level set everywhere in the triangle is a 1-manifold curve and topology of the level set stays the same. We will show that R contains no flat triangles, by contradiction.

Suppose there are flat triangles. Take a maximally edge-connected set of flat triangles T, so that no other flat triangles are edge-adjacent to any triangle in the set. Note that the boundary edges of T, which are shared by a single triangle in T, form closed loops. Consider the outmost loop of T, denoted as S. It is easy to show that S is a simple closed curve, i.e., each vertex is incident to two edges.

S cannot consist of only constrained vertices. If so, S would either consist of a subset of vertices on B, consist of vertices from B plus c, or be identical to B. The first case won't happen, as we require that no edge connects two non-consecutive vertices on B (see Section 5). The second case is clearly not possible, since the constraints are different at c and B, yet all vertices on S have the same value. The last case implies that all constraints on B are the same, which is only possible in the Cyclic Type. Since the center c has a different constraint (1) than that on B (0), the set T must contain a "hole" that excludes c. The boundary loop of this hole cannot be identical to or a subset of B, meaning there is some free vertex on this loop with value 0 but whose neighbors don't all have value 0. By Lemma 1, there must be some constraint less than 0, which contradicts with our definition of Cyclic Type.

As a result, S contains at least one free vertex. Note that the neighbors of a free vertex v on S do not all have the same value as v, and hence Lemma 1 applies there. We will reach contradiction by case enumeration on the number of free vertices in S:

1) Suppose S contains a single free vertex v. By Lemma 1, there should be constraints both above and below $w(v)$. On the other hand, it is easy to see that S contains at least three vertices. Therefore, there are at least two consecutive constrained vertices in S, implying they are also consecutive on B and whose constraints are identically $w(v)$, which contradicts with our first observation in Lemma 2.
2) Suppose S contains only two free vertices v_1, v_2. Since S contains at least three vertices, S must have at least one constrained vertex v, and the constraint there is $w(v_1)$. Applying Lemma 1 to v_1, there are constraints both above and below $w(v_1)$. Hence the field cannot be either Cyclic or Elliptic Type, where constraints have only two values (0,1).
Suppose now the field is one of the Flow Types. Consider the uphill and downhill paths originated from v_1, v_2. Note that none of these paths will intersect with S other than at v_1, v_2, since the values on these paths are strictly increasing or decreasing while the values on S are constant. Hence, if we apply a walk around the outside of S in counter-clockwise order starting from v, without loss of generality, we will encounter the two paths from v_1 and then the two paths from v_2 (see Figure 1 (a)). It is easy to check that this walk will either visit some downhill path

between two other uphill paths, or some uphill path between two other downhill paths, before coming back to v (Figure 1 (a) shows the first scenario). Now consider the ends of these paths, which will be constrained vertices on B (since c is absent). Since a downhill path from v_1 or v_2 cannot cross an uphill path from either vertex, and also because B is a simple closed curve like S, the order of "uphill" and "downhill" path-ends as we walk counter-clockwise on B are the same as how these paths originate on S. Therefore, there are four distinct vertices on B whose constraints compared to $w(v)$ have the pattern of either $\{0,-,+,-\}$, or $\{0,+,-,+\}$. This contradicts with our second observation in Lemma 2.

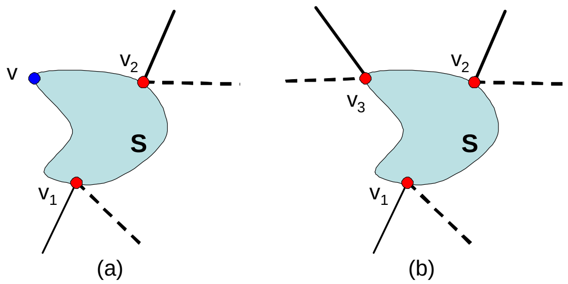

Fig. 1. Walking around S with two free vertices and one constrained vertex v (a), and with three free vertices (b). The uphill paths are drawn as solid lines, and downhill paths as dotted lines.

3) Suppose S contains at least three free vertices v_1,v_2,v_3. Consider the uphill and downhill paths originated from them. These paths will not intersect with S again, as argued above. It can be verified that, if we apply a walk around the outside of S in counter-clockwise order, we will always encounter four paths in sequence that alternate between being uphill and downhill, although they may not be consecutive (Figure 1 (b) shows one example). If these paths all end on the boundary B, then using the same argument above, there will be four distinct vertices on B whose constraints compared to $w(v_1)$ have an alternating pattern $\{+,-,+,-\}$ as we walk around B, which contradicts with Lemma 2. The remaining possibility is that the field is Cyclic Type and some of the four paths end at the center c. Since these paths cannot cross each other, as we argued above, at most one path can end at c. Hence at least one uphill path and one downhill path will end on B, contradicting with our definition in Cyclic Type that all constraints on B have the same value.

On an edge: A critical point exists on an interior edge $\{v_1,v_2\}$ only if $w(v_1)=w(v_2)$ and the two opposite vertices v_3,v_4 in the triangles sharing the edge have values that are both no less (or both no greater) than $w(v_1)$. Otherwise, the level set is well-defined along the edge with no topology-changing event. Since we already show that flat triangles don't exist, we only need to show that $w(v_3),w(v_4)$ cannot be both less (or both greater) than $w(v_1)$, which we do by contradiction.

Suppose both $w(v_3),w(v_4)$ are greater than $w(v_1)$ (the less-than case is similar). For each of these four vertices v_i, we will find a constrained vertex u_i as follows. If v_i is already constrained, then let u_i be v_i. Otherwise, let u_i be the end of the downhill path from v_i for $i=1,2$, and the end of the uphill path from v_i for $i=3,4$. Note that these downhill and uphill paths do not cross each other, as they have disparate range of values. For Elliptic and Flow Types, all u_i are on B, and u_1,u_2 will be separated by u_3,u_4 as we walk along B. By our hypothesis, u_1,u_2 have values that are strictly smaller than those of u_3,u_4, which contradicts with Lemma 2. Otherwise, suppose the field is of Cyclic Type. Since paths don't cross, at most one of the u_i can be at c, and hence at least one of u_1,u_2 and one of u_3,u_4 will be on B. This would imply that B has non-constant constraint values, which contradicts with our definition of Cyclic Type.

At a vertex: Any interior vertex that is not c is a free vertex. To ensure that a free vertex v is not a critical point, the following conditions need to be met. Let $P(v),N(v)$ denote the set of neighboring vertices of v whose values are greater or less than $w(v)$, respectively. Then, 1) both $P(v),N(v)$ should be non-empty, and should contain a single edge-connected component, and 2) a neighboring vertex with the same value as v may only exist between a vertex in $P(v)$ and a vertex in $N(v)$. Note that the second condition is already guaranteed by the previous case. We will next show that the first condition is met at each free vertex.

First, neither $P(v)$ or $N(v)$ can be empty. If so, either all neighbors of v have the same value as v, which contradicts our no-flat-triangle argument above, or $w(v)$ would be the minimum or maximum among its neighbors, which is not possible given the positivity of the mean value weights.

Next, suppose $N(v)$ (or similarly $P(v)$) contains two separate set of vertices S_1,S_2 such that no edges exist between the vertices in the two sets. Then there must be at least two separate sets S_3,S_4 in $P(v)$ so that a walk in the 1-ring of v will encounter these sets in the order S_1,S_3,S_2,S_4. Take one vertex v_i from each set S_i, we have $w(v_1),w(v_2)$ both strictly smaller than $w(v_3),w(v_4)$. Following the same argument in the edge case, a contradiction can be reached.

Finally, we show that the center c in Cyclic Type lies at a local maxima, which is a critical point. It is sufficient to show that every neighbor v of c has a value less than $w(c)$. This is obviously true if v lies on B. Otherwise, v is a free vertex, and by Lemma 1, it cannot have a value equal to or greater than $w(c)$, because otherwise there should be some constraint that is greater than $w(c)$, which contradicts with our definition. □

B.2 Boundary

Since a point on the boundary does not have a complete neighborhood, we cannot unambiguously characterize one as critical or regular. Instead, we will study the behavior of the level set at these points. The result below shows that the behavior is similar to the continuous case: the level set meets tangentially at most part of the boundary except near the terminals and along the flow-out segments. In addition, the level set is well-defined and manifold everywhere except at the terminal in Elliptic Type.

Proposition 3: The level set of $w(x)$ has the following topology at $x \in B$:

1) A single point, if x is at the terminal vertex in Elliptic Type.
2) The end of a 1-manifold curve, if x is on an edge next to the terminals in Elliptic Type and Flow Types, an edge contained in the flow-out segments in Flow Types, or at an vertex of such edges.
3) A 1-manifold curve that coincides with B, if x is at any other location of B.

Proof: We consider each case separately:

1) Using a similar argument as in the case of the center c in Cyclic Type, the terminal t in Elliptic Type has to be surrounded by vertices with less value than $w(t)$, which leads to the single-point level set.
2) Since any edge next to a terminal vertex or in the flow-out segments have non-equal values at their ends, the level curves on the single triangle sharing the edge are well-defined and end on the edge. Now, consider a vertex v of such edge. By our definition of constraints, the two neighboring constrained vertices v_1, v_2 do not have values that are both greater than, less than, or equal to that of v. Due to symmetry, we only need to consider two cases: 1) $w(v_1) > w(v) > w(v_2)$. Using a similar argument to that at an interior vertex, we can show that both $P(v)$ and $N(v)$ contains a single edge-connected component, and that at most one neighbor of v has equal value with v and that neighbor lies between the two sets $N(v)$ and $P(v)$ in the 1-ring of v. Hence there is only one level curve coming to v. 2) $w(v_1) \geqslant w(v) > w(v_2)$. We can derive that $P(v)$ is empty, $N(v)$ contains a single connected component, and that v_1 is the only neighbor with the same value as v. Hence there is only one level curve coming to v (following the edge $\{v_1, v\}$).
3) The remaining edges of B have equal values at their ends. Since there is no flat triangle, the level curve is well-defined along each edge and coincide with the edge. Now, consider a vertex v shared by two such edges. Note that, besides two constrained vertices v_1, v_2 whose constraints are the same as $w(v)$, other neighbors of v (which are free vertices) have values that are either all less than or all greater than $w(v)$. Otherwise, by Lemma 1, there will be constraints both above and below $w(v)$, which contradicts with Lemma 2. As a result, the only level curve at v is along the edges $\{v, v_1\}$ and $\{v, v_2\}$. $\square$

References

[1] J. L. Walsh. *The Location of Critical Points of Analytic and Harmonic Functions*. American Mathematical Society, New York, USA, 1950.