Washington University in St. Louis
School of Engineering & Applied Science

CSE 554

Lecture 8: Laplacian Deformation

Fall 2012

Review

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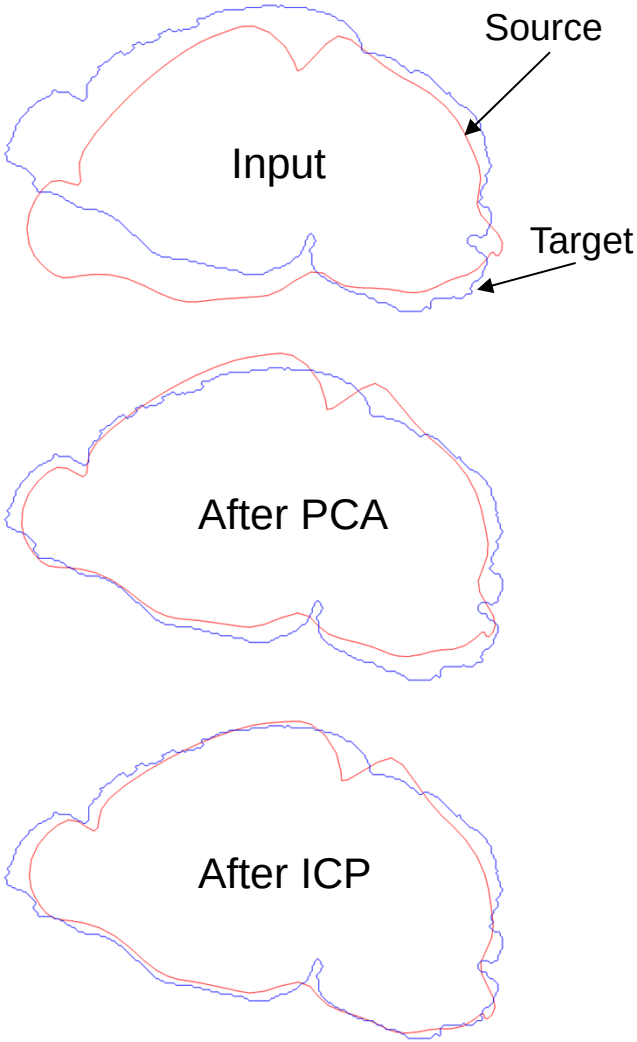
- Alignment
  - Registering source to target by rotation and translation
    - Rigid-body transformations
- Methods
  - Aligning principle directions (PCA)
  - Aligning corresponding points (SVD)
  - Iterative improvement (ICP)
    - Combines PCA and SVD



Non-rigid Registration

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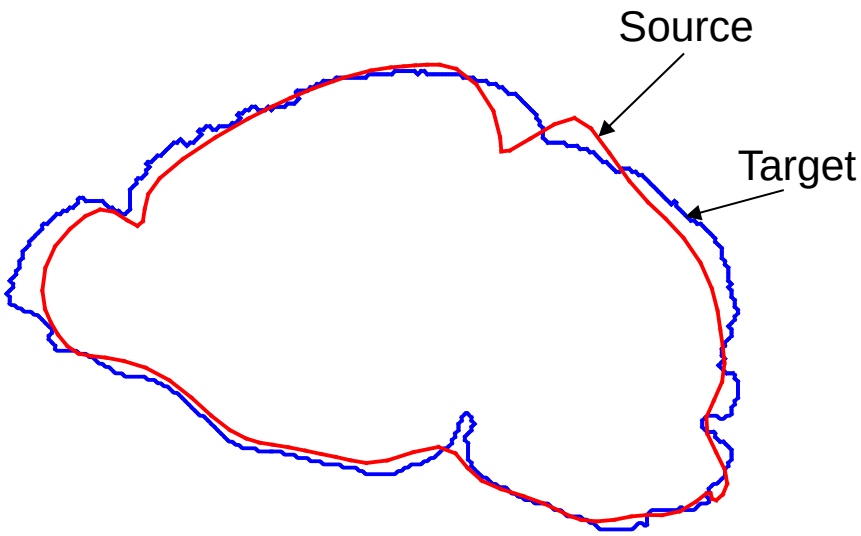
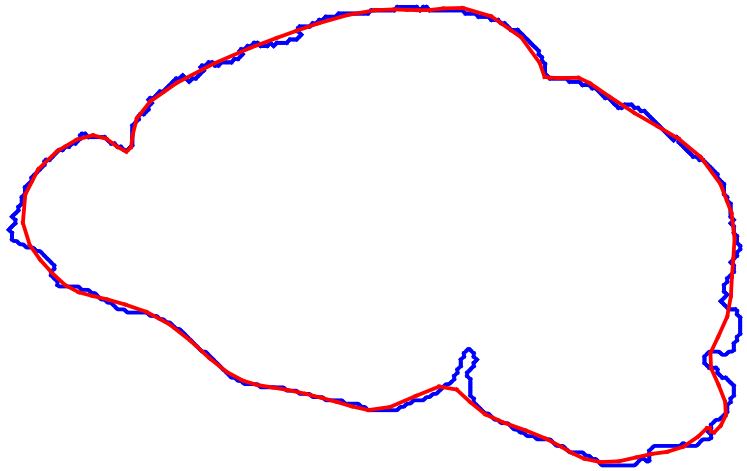
- Rigid alignment cannot account for shape variance
- Non-rigid deformation can give a better fit



Rigid alignment



After non-rigid deformation

Non-rigid Registration

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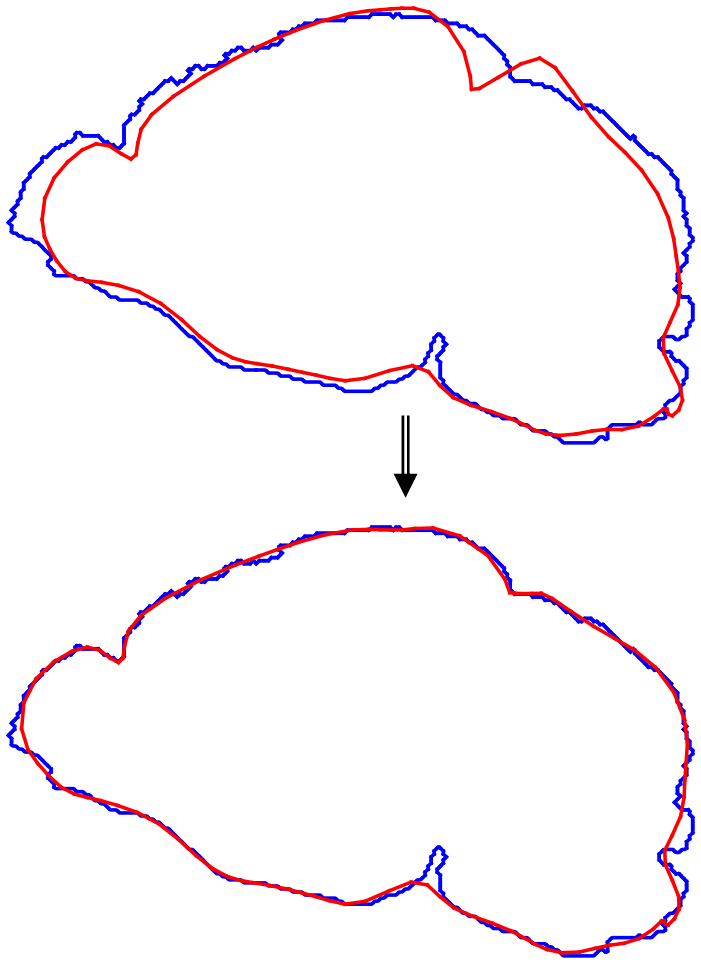
- A minimization problem
  - Minimizing the distance between the deformed source and the target
    - "Fitting term"
  - Minimizing the distortion to the source shape
    - "Distortion term"



Intrinsic vs. Extrinsic

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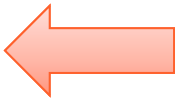
- Intrinsic methods
  - Deforms points on the source curve/surface
  - App: boundary curve or surface matching



- Extrinsic methods
  - Deforms all points on and interior to the source curve/surface
  - App: image or volume matching

Laplacian-based Deformation

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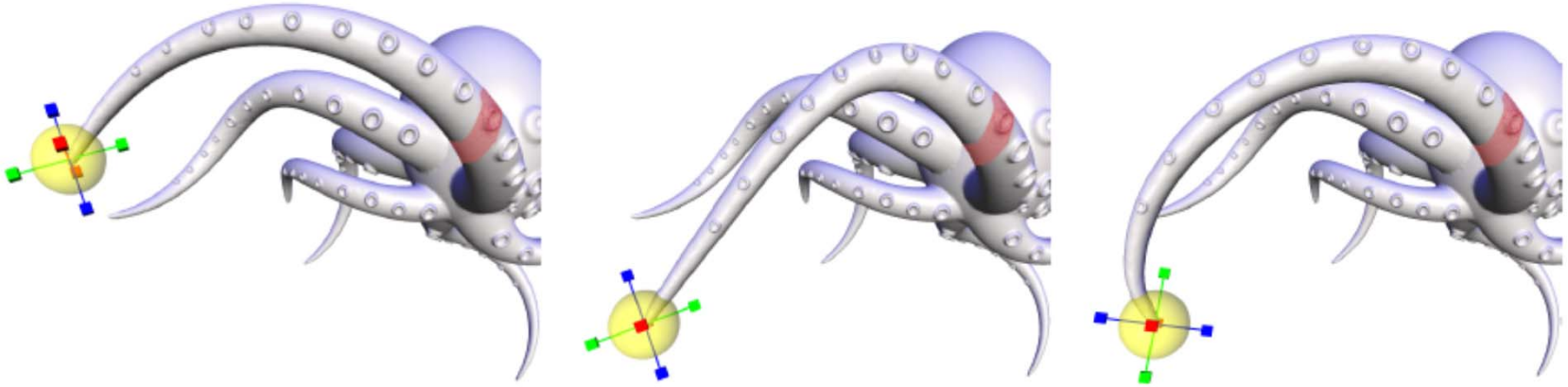
- An intrinsic method
  - Simple, efficient, producing reasonable results
    - Preserving local shape features
  - Widely used in graphics applications for interactive deformation



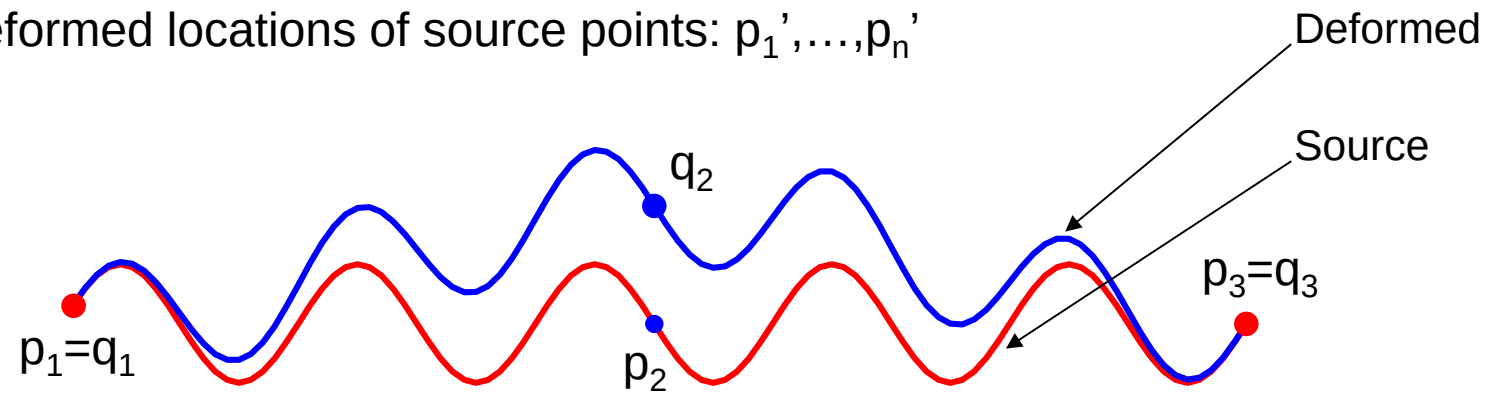
Reference: “Laplacian surface editing”, by Sorkine et al., 2004 (citation ~ 500)

Setup

- Input
  - Source with n points: $p_1,\ldots,p_n$
    - Let the first m points be "handles"
  - Target location of handles: $q_1,\ldots,q_m$

When deforming the source to fit a target shape: m=n and q_i is the point on the target closest to p_i.

- Output
  - Deformed locations of source points: $p_1',\ldots,p_n'$



An example with 3 target points, two of which are stationary (red)

Overview

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- Finding deformed locations p_i' that minimize:

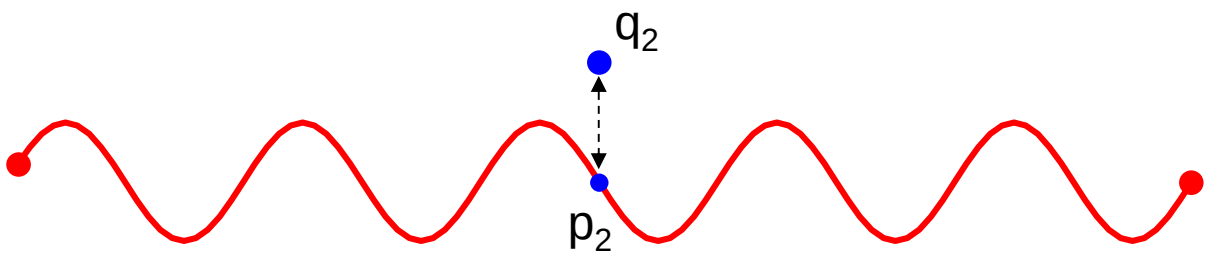
$$E = E_f + E_d$$

  - E_f: fitting term
    - Measures how close are the deformed source to the target
  - E_d: distortion term
    - Measures how much the source shape is changed

Fitting Term

- Sum of squared distances to target handle locations

$$E_f = \sum_{i=1}^{m} \| p_i' - q_i \|^2$$



Distortion Term

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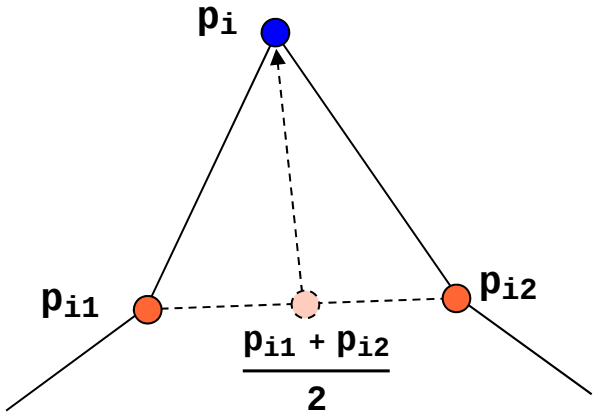
- Q: How to measure shape?
- A: By "bumpiness" at each vertex
  - Laplacian: vector from the centroid of neighbors to the vertex
    - Recall that in fairing, we reduced this vector to "smooth out" bumps
    - A linear operator over point locations

$$L[p_i] = p_i - \frac{1}{|N_i|} \sum_{j \in N_i} p_j$$

where N_i are indices of neighboring vertices of p_i



Distortion Term

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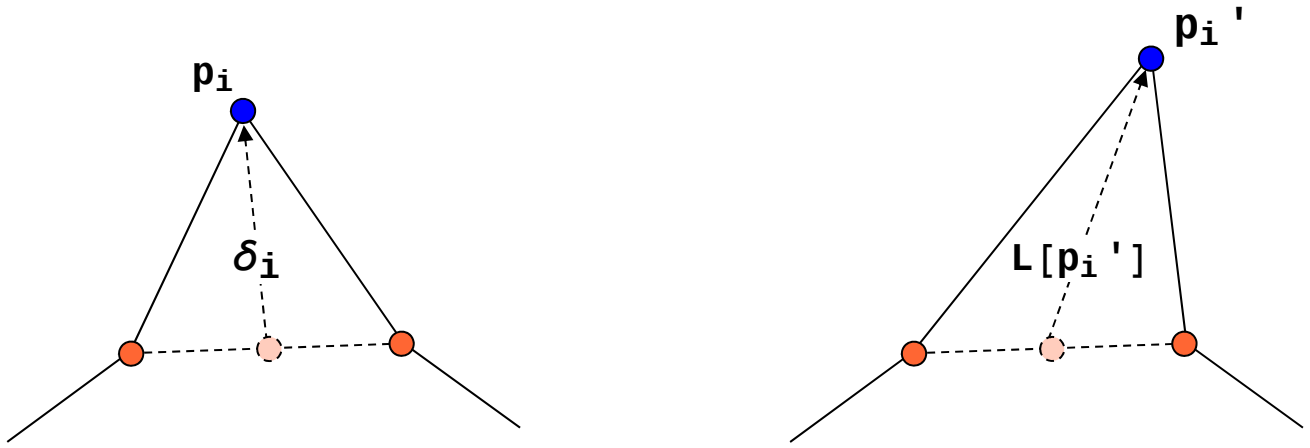
- Minimizing changes in Laplacians during deformation
  - Over all source points

$$E_d = \sum_{i=1}^{n} \| L[p_i'] - \delta_i \|^2$$

δ_i: Laplacian at p_i before deformation



Putting Together

- Finding deformed locations p_i' that minimize:

$$E = E_f + E_d$$

$$= \sum_{i=1}^{m} \|p_i' - q_i\|^2 + \sum_{i=1}^{n} \|L[p_i'] - \delta_i\|^2$$

  - A quadratic equation in terms of variables (p_{ix}', p_{iy}', p_{iz}')
    - q_i, δ_i are constants
    - **L**[] is a linear operator

Quadratic Minimization

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- A general form of quadratic minimization:

$$\min \sum_{i=1}^{k} \left\| a_i^T x - b_i \right\|^2$$

  - There are s variables: $x=(x_1,\ldots,x_s)^T$
  - Each $a_1,\ldots, a_k$ is a length-s column vector (linear coefficients)
  - Each $b_1,\ldots, b_k$ is a scalar (constant coefficients)
  - k should be greater than s (so that the problem is over-constrained)

Quadratic Minimization

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- To solve:

$$\min \sum_{i=1}^{k} \left\| a_i^T x - b_i \right\|^2$$

- Re-write in matrix form:

$$\min \|A x - B\|^2$$

where $A = \begin{pmatrix} a_1^T \\ \vdots \\ a_k^T \end{pmatrix}$ is a k by s matrix

$B = \begin{pmatrix} b_1 \\ \vdots \\ b_k \end{pmatrix}$ is a length-k vector

Quadratic Minimization

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- The minimizer is where the partial derivatives are all zero

$$0 = \frac{\partial \|Ax - B\|^2}{\partial x}$$
$$= 2A^T A x - 2A^T B \quad \Longrightarrow \quad A^T A x = A^T B$$

  - To solve for x in this equation:
    - Taking matrix inverse (good for small s, but numerically unstable for large s)

$$x = \left(A^T A\right)^{-1} A^T B$$

    - Using specialized linear system solver (LinearSolve in *Mathematica*, TNT/LAPACK in C)

Quadratic Minimization

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- Re-writing our minimization in the general form

$$E = E_f + E_d$$

$$= \sum_{i=1}^{m} \|p_i' - q_i\|^2 + \sum_{i=1}^{n} \|L[p_i'] - \delta_i\|^2$$

  - In 2D, there are 2n variables: $x = (p_{1x}', \ldots, p_{nx}', p_{1y}', \ldots, p_{ny}')^T$
    - In 3D, there are 3n variables
  - We will next re-write each quadratic term in 2D as $(a_i x - b_i)^2$
    - Can be extended easily to 3D

Quadratic Minimization

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- The a_i and b_i in the fitting term

$$E_f = \sum_{i=1}^{m} \|p_i' - q_i\|^2 = \sum_{i=1}^{m} (p_{ix}' - q_{ix})^2 + \sum_{i=1}^{m} (p_{iy}' - q_{iy})^2$$

  - There are 2m quadratic terms

$$\sum_{i=1}^{2m} (a_i^T x - b_i)^2$$

$$x = (p_{1x}', \ldots, p_{nx}', p_{2y}', \ldots, p_{ny}')$$

  - In the first set of m terms:
    - For i=1,…,m, $b_i=q_{ix}$, a_i contains all zero, except its (i)th entry is 1.
  - In the second set of m terms:
    - For i=1,…,m, $b_{i+m}=q_{iy}$, a_{i+m} contains all zero, except its (i+n)th entry is 1

Quadratic Minimization

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- The a_i and b_i in the fitting term

$$E_f = \sum_{i=1}^{m} \|p_i' - q_i\|^2 = \sum_{i=1}^{m} (p_{ix}' - q_{ix})^2 + \sum_{i=1}^{m} (p_{iy}' - q_{iy})^2$$

  - There are 2m quadratic terms

$$\sum_{i=1}^{2m} (a_i^T x - b_i)^2$$

  - Example with 3 vertices and 2 fitting constraints (n=3; m=2):

$$a_1^T = (1\ 0\ 0\ 0\ 0\ 0)$$
$$a_2^T = (0\ 1\ 0\ 0\ 0\ 0)$$
$$a_3^T = (0\ 0\ 0\ 1\ 0\ 0)$$
$$a_4^T = (0\ 0\ 0\ 0\ 1\ 0)$$

$$x = \begin{pmatrix} p_{1x}' \\ p_{2x}' \\ p_{3x}' \\ p_{1y}' \\ p_{2y}' \\ p_{3y}' \end{pmatrix}$$

$$b_1 = q_{1x}$$
$$b_2 = q_{2x}$$
$$b_3 = q_{1y}$$
$$b_4 = q_{2y}$$

Quadratic Minimization

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- The a_i and b_i in the distortion term:

$$L[p_i] = p_i - \frac{1}{|N_i|}\sum_{j \in N_i} p_j$$

$$E_d = \sum_{i=1}^{n} \|L[p_i'] - \delta_i\|^2 = \sum_{i=1}^{n} (L[p_{ix}'] - \delta_{ix})^2 + \sum_{i=1}^{n} (L[p_{iy}'] - \delta_{iy})^2$$

  - There are 2n quadratic terms $\sum_{i=1}^{2n} (a_i^T x - b_i)^2$
  - The first set of n terms:
    - For i=1,…,n, a_i is all zero except the (i)th entry is 1, the (j)th entries are $-1/|N_i|$ for all $j \in N_i$, and $b_i = \delta_{ix}$
  - The second set of n terms:
    - For i=1,…,n, a_{i+n} is all zero except the (i+n)th entry is 1, the (j+n)th entries are $-1/|N_i|$ for all $j \in N_i$, and $b_{i+n} = \delta_{iy}$

Quadratic Minimization

- The a_i and b_i in the distortion term:

$$L[p_i] = p_i - \frac{1}{|N_i|}\sum_{j \in N_i} p_j$$

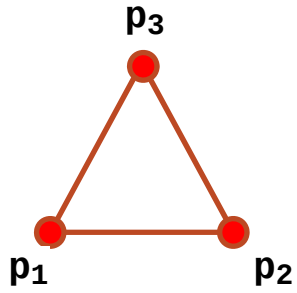
$$E_d = \sum_{i=1}^{n} \|L[p_i'] - \delta_i\|^2 = \sum_{i=1}^{n} (L[p_{ix}'] - \delta_{ix})^2 + \sum_{i=1}^{n} (L[p_{iy}'] - \delta_{iy})^2$$

  - There are 2n quadratic terms $\sum_{i=1}^{2n} (a_i^T x - b_i)^2$
  - Example with 3 vertices (n=3):

$$a_1^T = \left(1 \;\; -\tfrac{1}{2} \;\; -\tfrac{1}{2} \;\; 0 \;\; 0 \;\; 0\right)$$
$$a_2^T = \left(-\tfrac{1}{2} \;\; 1 \;\; -\tfrac{1}{2} \;\; 0 \;\; 0 \;\; 0\right)$$
$$a_3^T = \left(-\tfrac{1}{2} \;\; -\tfrac{1}{2} \;\; 1 \;\; 0 \;\; 0 \;\; 0\right)$$
$$a_4^T = \left(0 \;\; 0 \;\; 0 \;\; 1 \;\; -\tfrac{1}{2} \;\; -\tfrac{1}{2}\right)$$
$$a_5^T = \left(0 \;\; 0 \;\; 0 \;\; -\tfrac{1}{2} \;\; 1 \;\; -\tfrac{1}{2}\right)$$
$$a_6^T = \left(0 \;\; 0 \;\; 0 \;\; -\tfrac{1}{2} \;\; -\tfrac{1}{2} \;\; 1\right)$$

$$x = \begin{pmatrix} p_{1x}' \\ p_{2x}' \\ p_{3x}' \\ p_{1y}' \\ p_{2y}' \\ p_{3y}' \end{pmatrix}$$

$$b_1 = \delta_{1x}$$
$$b_2 = \delta_{2x}$$
$$b_3 = \delta_{3x}$$
$$b_4 = \delta_{1y}$$
$$b_5 = \delta_{2y}$$
$$b_6 = \delta_{3y}$$



Summary

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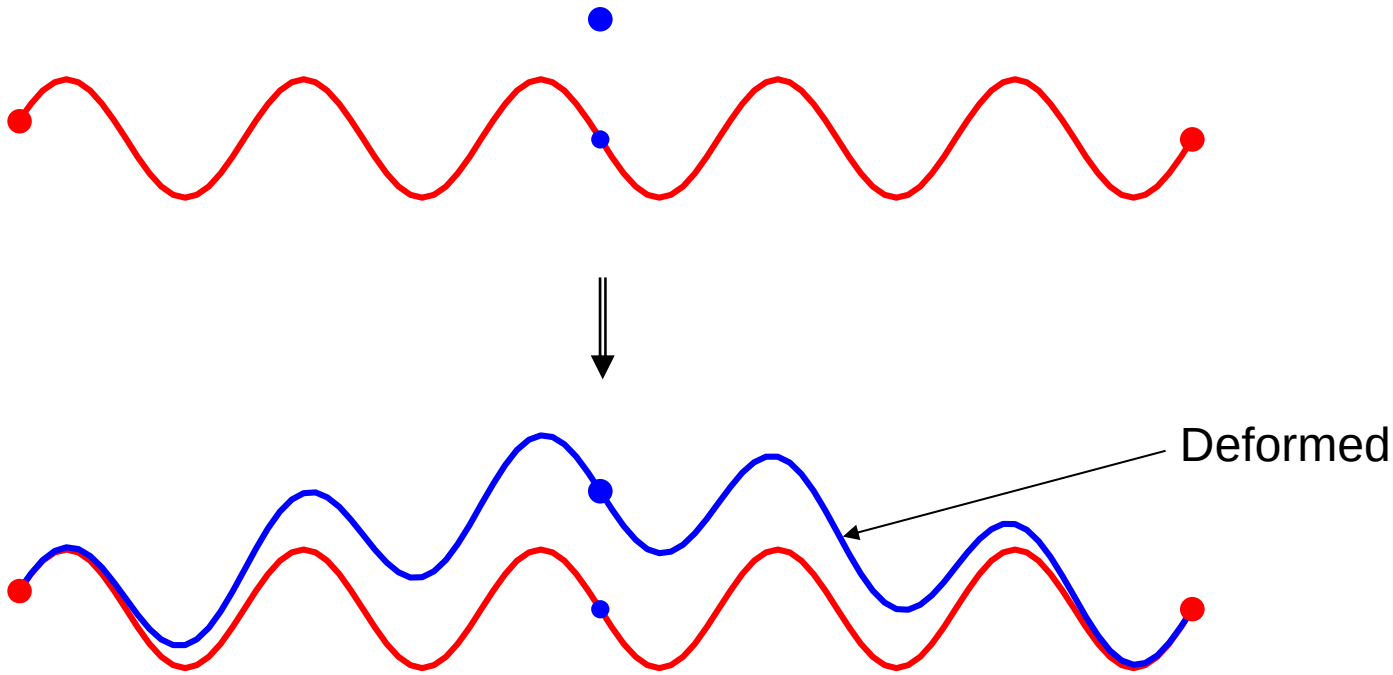
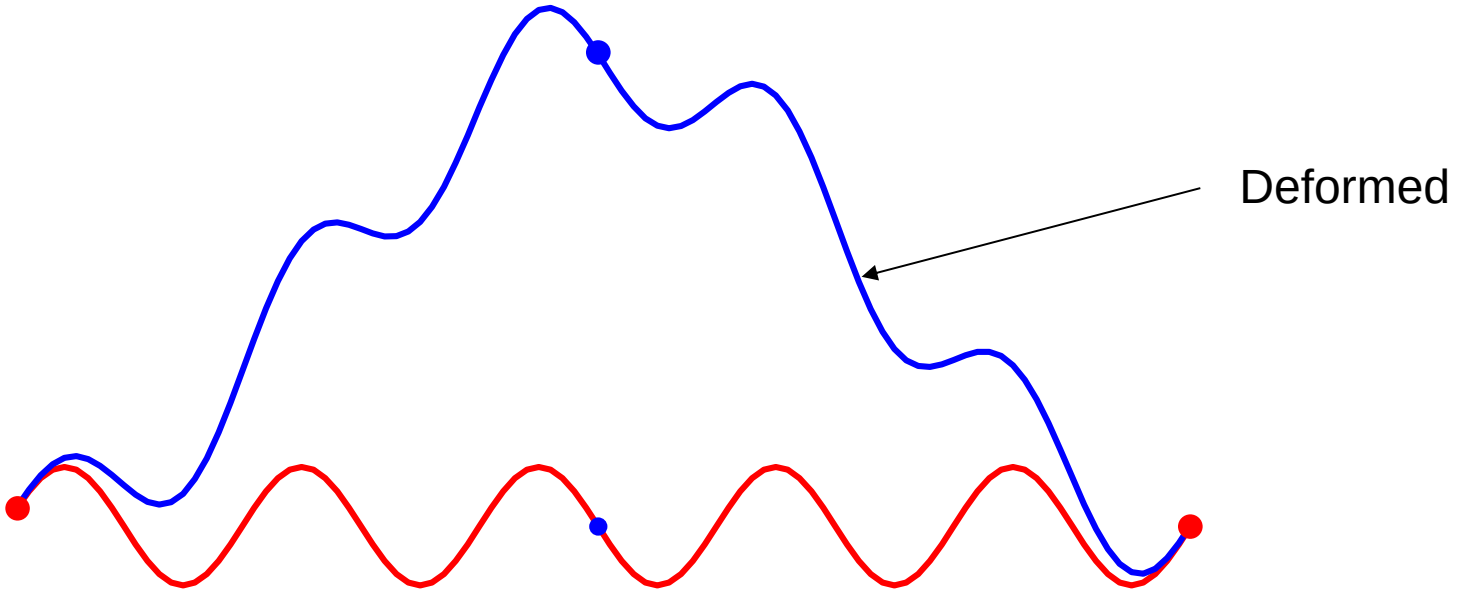
- Compute Laplacians (δ_i)
- Construct coefficients (a_i, b_i)
  - Stick them into matrices (A,B)
- Solve (x)
  - $\mathbf{A^T A x = A^T B}$

Results



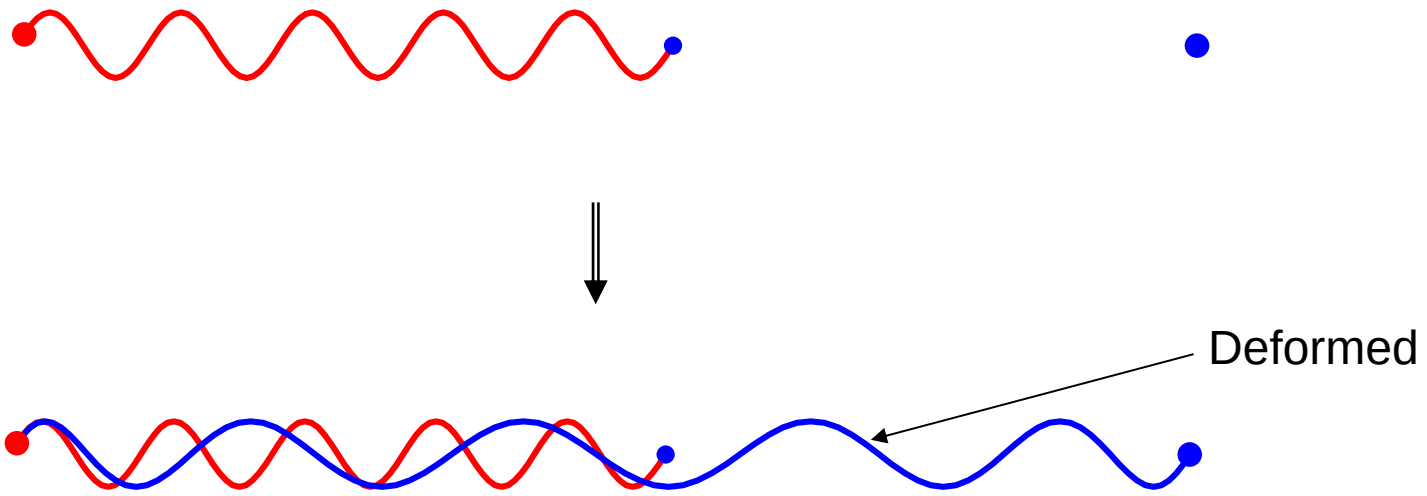
A small deformation

Results



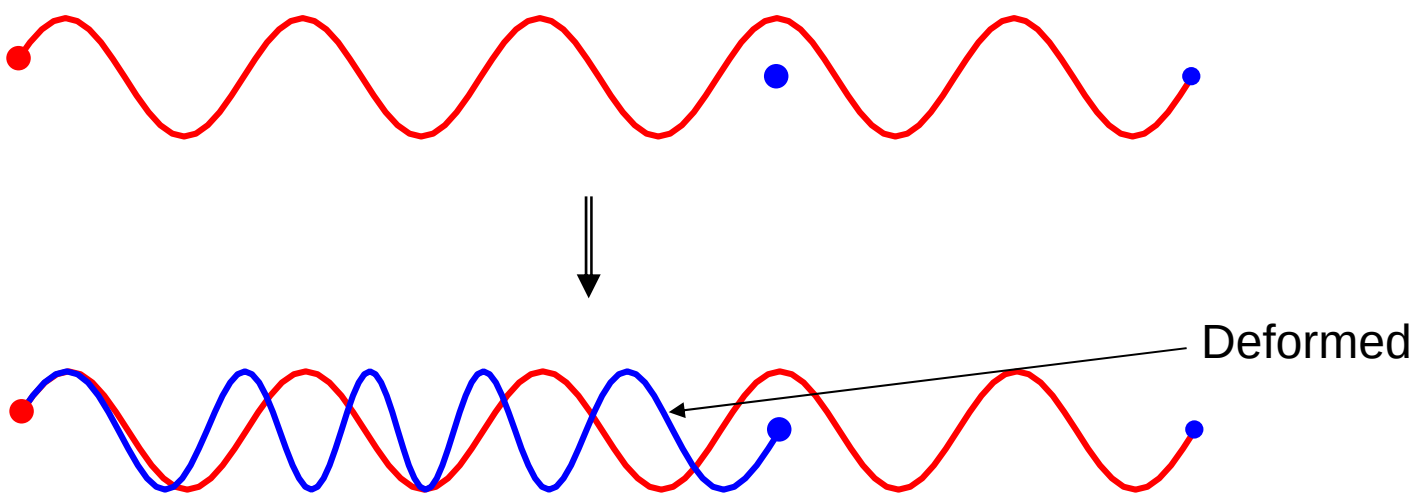
A larger deformation

Results



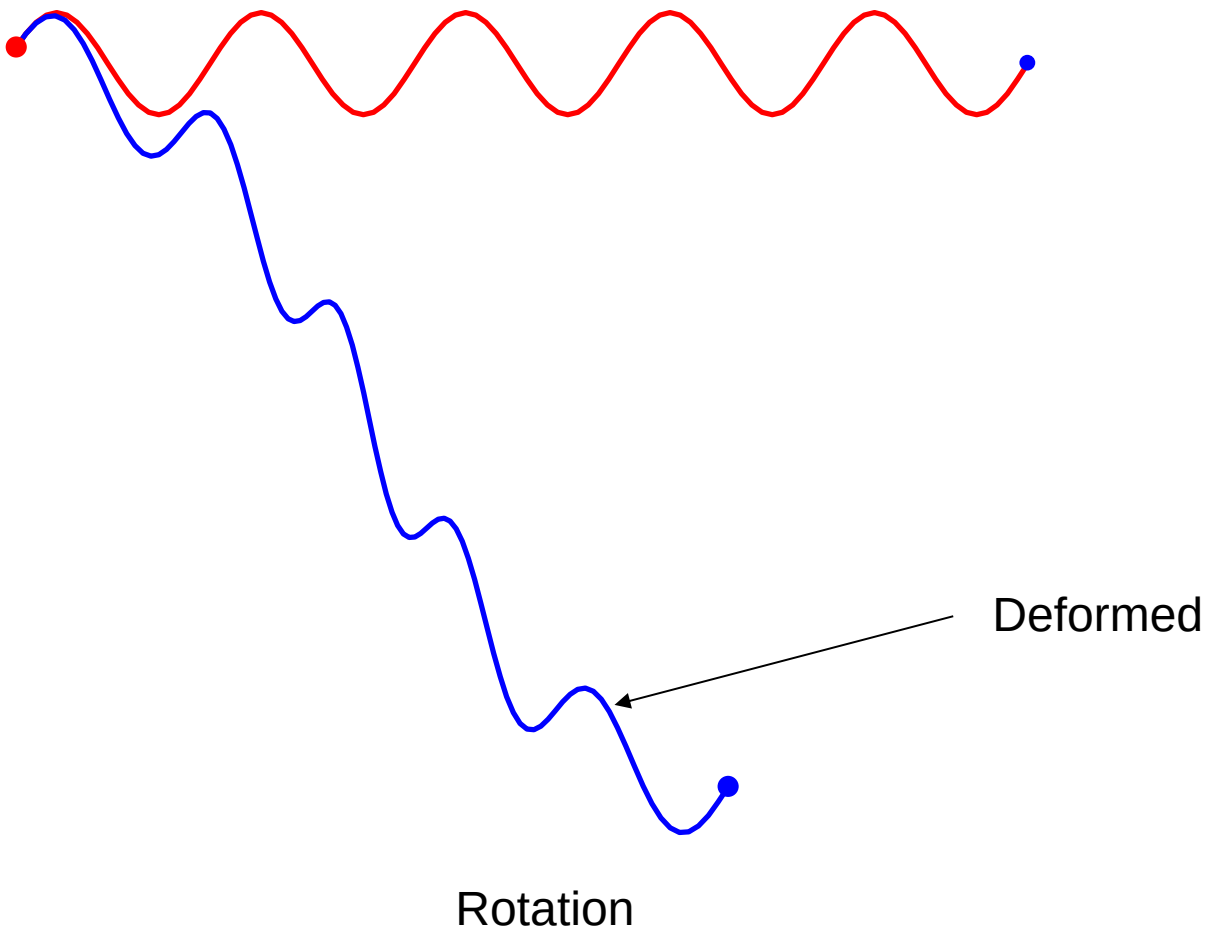
Stretching

Results



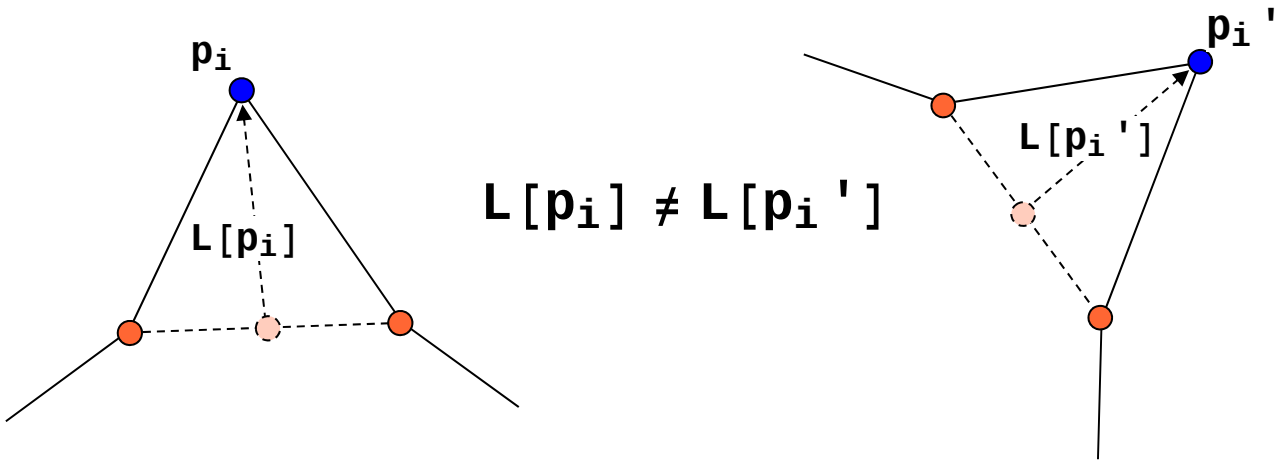
Shrinking

Results



Discussion

- Limitations
  - Local features are “skewed”, and they don’t scale with the model
- Reason: Laplacian changes with rotation or scale
  - Two bumps that differ by rotation or scale have different Laplacians
    - Which will be penalized by our distortion term

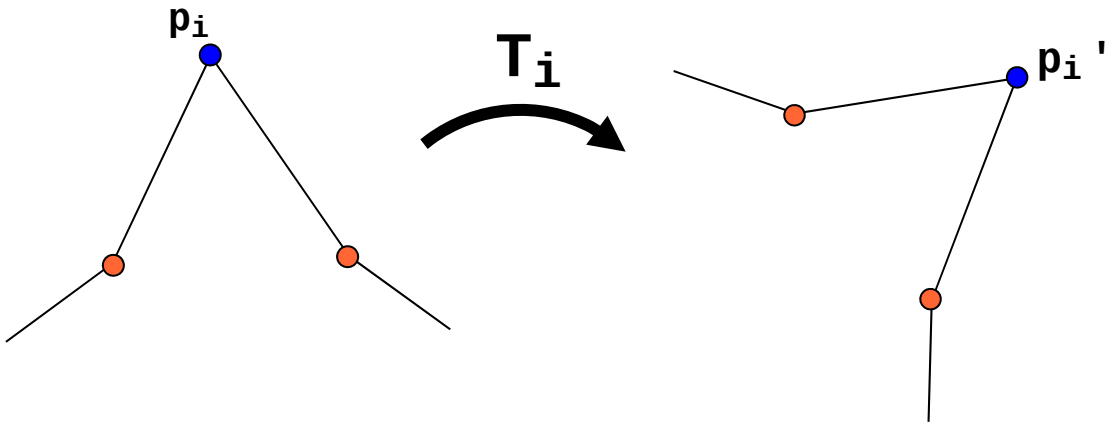


A Better Distortion Term

- Not penalizing rotation and scaling of local features
  - Transforming the original Laplacian vectors before comparing to the deformed Laplacians

$$E_d = \sum_{i=1}^{n} \| L[p_i'] - T_i \delta_i \|^2$$

- T_i is a matrix that describes how the local shape around p_i is deformed



Key Questions

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- How to represent transformations as matrices?
- How to compute T_i?
- We will focus in the derivations of the 2D case
  - 3D results will be briefly presented at the end

Transformation Matrices (2D)

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- Homogeneous coordinates
  - A 2D point: (x,y,1)
    - A 2D vector: (x,y,0)
  - A 3D point: (x,y,z,1)
    - A 3D vector: (x,y,z,0)

Transformation Matrices (2D)

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- Translation
  - Cartesian coordinates: vector addition

$$\begin{pmatrix} p_x' \\ p_y' \end{pmatrix} = \begin{pmatrix} v_x \\ v_y \end{pmatrix} + \begin{pmatrix} p_x \\ p_y \end{pmatrix}$$

  - Homogeneous coordinates: matrix product

$$\begin{pmatrix} p_x' \\ p_y' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & v_x \\ 0 & 1 & v_y \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} p_x \\ p_y \\ 1 \end{pmatrix}$$

Transformation Matrices (2D)

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- Isotropic scaling
  - Cartesian coordinates: vector scaling

$$\begin{pmatrix} p_x' \\ p_y' \end{pmatrix} = s * \begin{pmatrix} p_x \\ p_y \end{pmatrix}$$

  - Homogeneous coordinates: matrix product

$$\begin{pmatrix} p_x' \\ p_y' \\ 1 \end{pmatrix} = \begin{pmatrix} s & 0 & 1 \\ 0 & s & 1 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} p_x \\ p_y \\ 1 \end{pmatrix}$$

Transformation Matrices (2D)

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- Rotation
  - Cartesian coordinates: matrix product

$$\begin{pmatrix} p_x' \\ p_y' \end{pmatrix} = \begin{pmatrix} \mathrm{Cos}[\alpha] & -\mathrm{Sin}[\alpha] \\ \mathrm{Sin}[\alpha] & \mathrm{Cos}[\alpha] \end{pmatrix} \cdot \begin{pmatrix} p_x \\ p_y \end{pmatrix}$$

  - Homogeneous coordinates: matrix product

$$\begin{pmatrix} p_x' \\ p_y' \\ 1 \end{pmatrix} = \begin{pmatrix} \mathrm{Cos}[\alpha] & -\mathrm{Sin}[\alpha] & 0 \\ \mathrm{Sin}[\alpha] & \mathrm{Cos}[\alpha] & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} p_x \\ p_y \\ 1 \end{pmatrix}$$

Transformation Matrices (2D)

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- Summary of elementary similarity transformations
  - To combine transformations: take the product of these matrices

$$\begin{pmatrix} p_x' \\ p_y' \\ 1 \end{pmatrix} = M \cdot \begin{pmatrix} p_x \\ p_y \\ 1 \end{pmatrix}$$

$$\mathrm{Trs}[v] = \begin{pmatrix} 1 & 0 & v_x \\ 0 & 1 & v_y \\ 0 & 0 & 1 \end{pmatrix}$$ Translation by vector v

$$\mathrm{Scl}[s] = \begin{pmatrix} s & 0 & 1 \\ 0 & s & 1 \\ 0 & 0 & 1 \end{pmatrix}$$ Scaling by scalar s

$$\mathrm{Rot}[\alpha] = \begin{pmatrix} \mathrm{Cos}[\alpha] & -\mathrm{Sin}[\alpha] & 0 \\ \mathrm{Sin}[\alpha] & \mathrm{Cos}[\alpha] & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ Rotation by angle α

Similarity Transforms (2D)

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- General similarity transformations

$$T = \begin{pmatrix} a & w & t_x \\ -w & a & t_y \\ 0 & 0 & 1 \end{pmatrix}$$

  - The product of any set of elementary matrices can be written this way
  - Any choice of (a, w, t_x, t_y) can be written as a sequence of rotation, isotropic scaling and translation

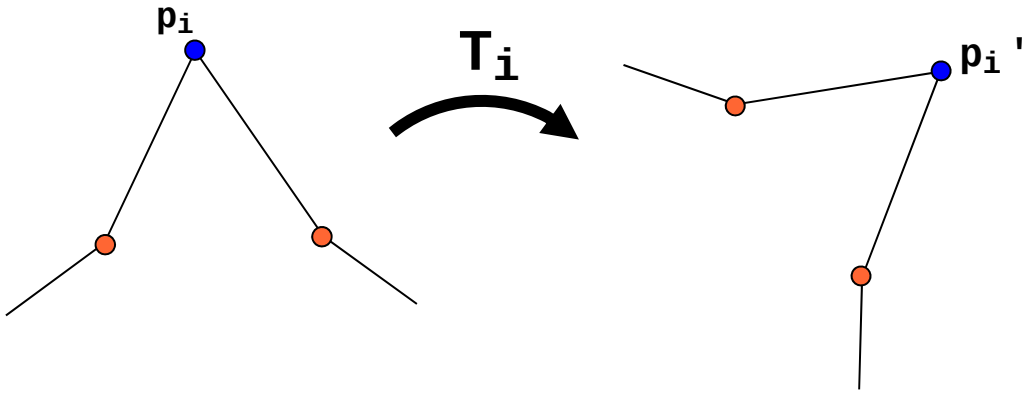
$$\begin{pmatrix} a & w & t_x \\ -w & a & t_y \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{Trs}[t_x, t_y].\mathbf{Scl}\left[\sqrt{a^2 + w^2}\right].\mathbf{Rot}\left[\mathbf{ArcTan}\left[\frac{a}{-w}\right]\right]$$

    - Note that a and w can't be both zero

Computing T_i (2D)

- Suppose we know the deformed locations p_i'
- Compute T_i as the similarity transform that best fits the neighborhood of p_i to that of p_i'

$$\min \left(\| T_i\, p_i - p_i' \|^2 + \sum_{j \in N_i} \| T_i\, p_j - p_j' \|^2 \right)$$



Computing T_i (2D)

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- Suppose we know the deformed locations p_i'
- Compute T_i as the similarity transform that best fits the neighborhood of p_i to that of p_i'

$$\min \left(\| T_i \, p_i - p_i' \|^2 + \sum_{j \in N_i} \| T_i \, p_j - p_j' \|^2 \right)$$

- This is a quadratic minimization problem for entries of T_i
  - E.g., a, w, t_x, t_y

Computing T_i (2D)

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- The matrix form of the minimization is:

$$\min \left\| C \begin{pmatrix} a \\ w \\ t_x \\ t_y \end{pmatrix} - \begin{pmatrix} p_{i_x}' \\ p_{i_y}' \\ p_{i1_x}' \\ p_{i1_y}' \\ \vdots \end{pmatrix} \right\|^2$$

where $C = \begin{pmatrix} p_{i_x} & p_{i_y} & 1 & 0 \\ p_{i_y} & -p_{i_x} & 0 & 1 \\ p_{i1_x} & p_{i1_y} & 1 & 0 \\ p_{i1_y} & -p_{i1_x} & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}$ is a $2|N_i|+2$ by 4 matrix, and N_i={i1, i2,…} are indices of neighboring vertices of p_i

Computing T_i (2D)

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- By quadratic minimization:

$$\begin{pmatrix} a \\ w \\ t_x \\ t_y \end{pmatrix} = \left(C^T C\right)^{-1} C^T \begin{pmatrix} p_{ix}' \\ p_{iy}' \\ p_{i1x}' \\ p_{i1y}' \\ \vdots \end{pmatrix}$$

  - Linear expressions of variables (p_{ix}' , p_{iy}')

Distortion Term (2D)

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- Two parts of each distortion term: $\|\mathbf{L}[\mathbf{p}_i'] - \mathbf{T}_i\,\boldsymbol{\delta}_i\|^2$
  - Transformed Laplacian:

$$\mathbf{T}_i\,\boldsymbol{\delta}_i = \mathbf{D}\begin{pmatrix} a \\ w \\ t_x \\ t_y \end{pmatrix} = \mathbf{D}\left(\mathbf{C}^T\mathbf{C}\right)^{-1}\mathbf{C}^T\begin{pmatrix} p_{i_x}' \\ p_{i_y}' \\ p_{i1_x}' \\ p_{i1_y}' \\ \vdots \end{pmatrix} \quad \text{where } \mathbf{D} = \begin{pmatrix} \delta_{i_x} & \delta_{i_y} & 0 & 0 \\ \delta_{i_y} & -\delta_{i_x} & 0 & 0 \end{pmatrix}$$

  - Laplacian of the deformed locations:

$$\mathbf{L}[\mathbf{p}_i'] = \mathbf{L}\begin{pmatrix} p_{i_x}' \\ p_{i_y}' \\ p_{i1_x}' \\ p_{i1_y}' \\ \vdots \end{pmatrix} \quad \text{where } \mathbf{L} = \begin{pmatrix} 1 & 0 & -\frac{1}{|N_i|} & 0 & \cdots \\ 0 & 1 & 0 & -\frac{1}{|N_i|} & \cdots \end{pmatrix}$$

is a 2 by $2|N_i|+2$ matrix

Distortion Term (2D)

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- Putting together:

$$E_d = \sum_{i=1}^{n} \| L[p_i'] - T_i \delta_i \|^2$$

$$= \sum_{i=1}^{n} \left(H[\![1]\!] \begin{pmatrix} p_{ix}' \\ p_{iy}' \\ p_{i1x}' \\ p_{i1y}' \\ \vdots \end{pmatrix} \right)^2 + \sum_{i=1}^{n} \left(H[\![2]\!] \begin{pmatrix} p_{ix}' \\ p_{iy}' \\ p_{i1x}' \\ p_{i1y}' \\ \vdots \end{pmatrix} \right)^2$$

where $H = L - D \left(C^T C\right)^{-1} C^T$

and $H[\![1]\!]$, $H[\![2]\!]$ are its rows

  - They form 2n quadratic terms $(a_i x - b_i)^2$ for $x = (p_{1x}', \ldots, p_{nx}', p_{1y}', \ldots, p_{ny}')^T$
    - All b_i are zero
    - Each a_i can be extracted from H

Results (2D)

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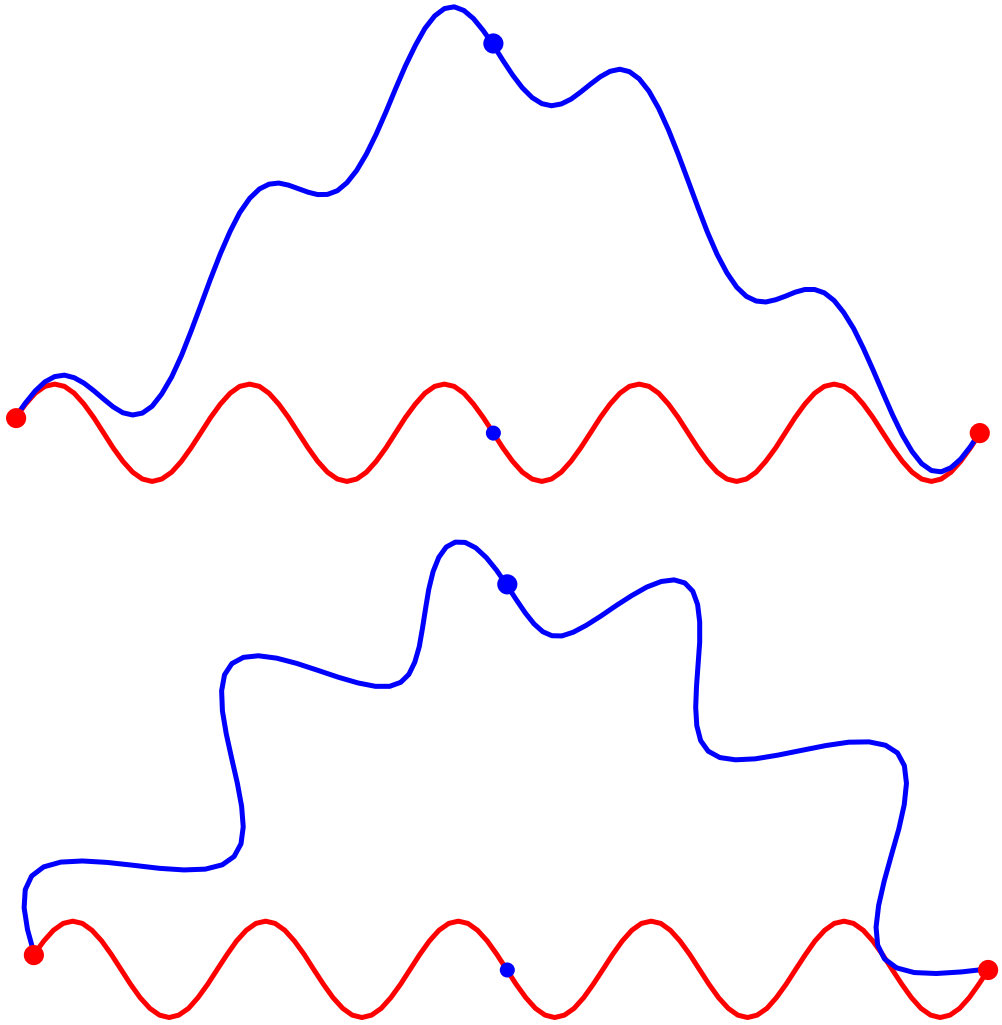
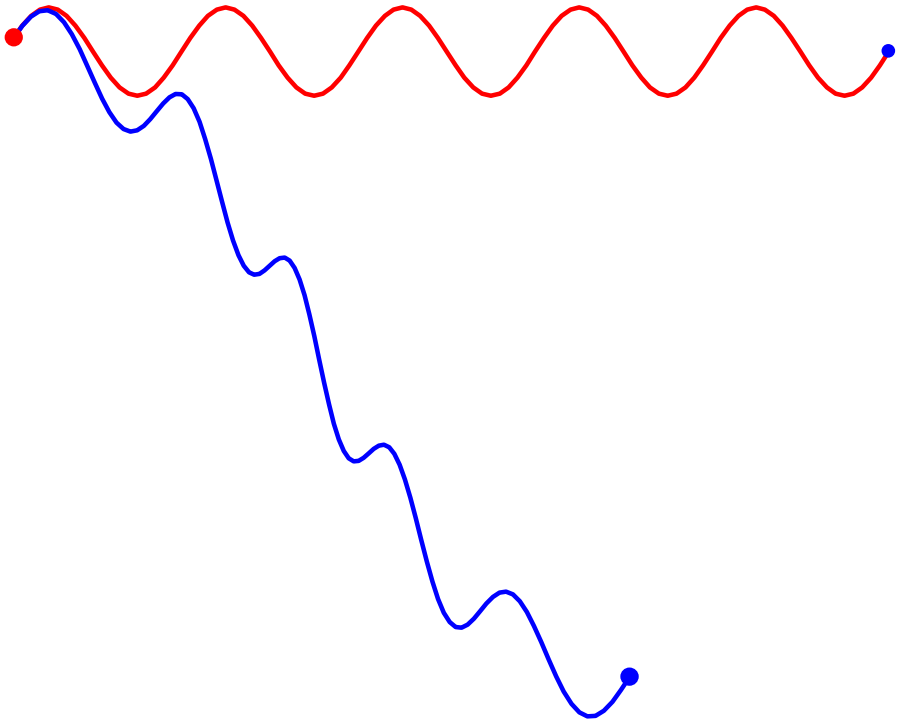


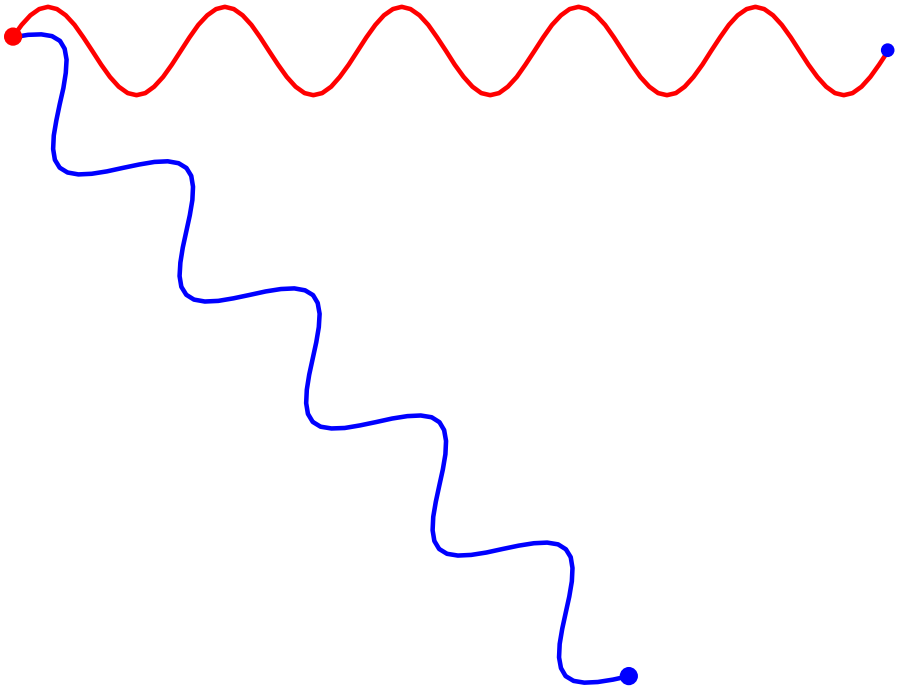
Old distortion term

New distortion term

Results (2D)



Old distortion term



New distortion term

Results (2D)

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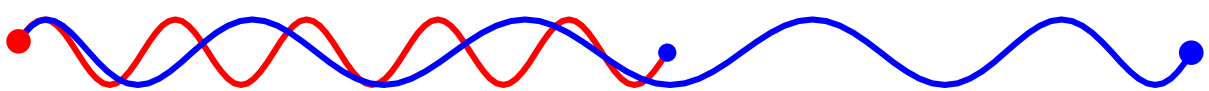
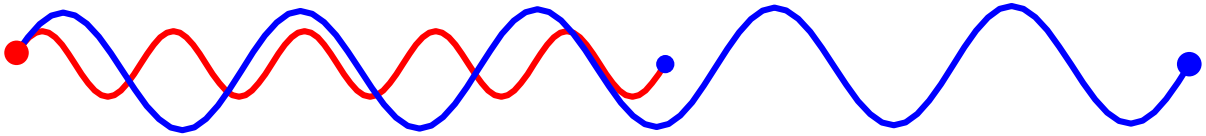


Old distortion term



New distortion term

Results (2D)

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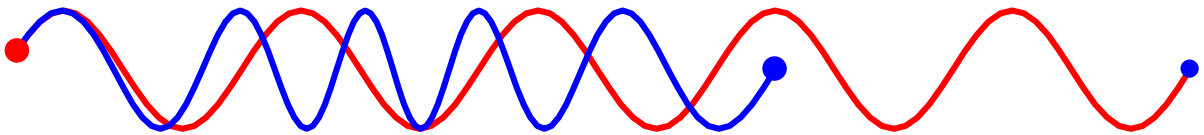
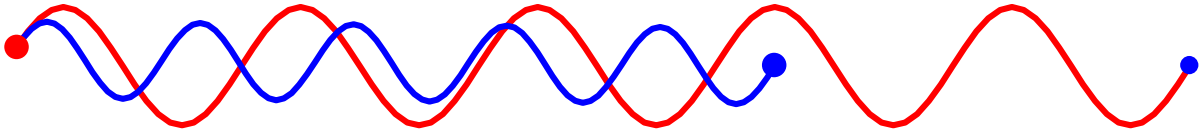


Old distortion term



New distortion term

Registration

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- Use nearest neighbors as corresponding target locations
  - Assuming the source is already close to the target

- Iterative closest point (ICP)
  - 1. For each point on the source, assign its closest point on the target as its corresponding point. Compute Laplacian-based deformation.
    - A threshold on the closest distance can be used to throw away unlikely correspondences
  - 2. Repeat step (1) until a termination criteria is met.
    - Maximum iteration or minimum RMSD improvement

Result

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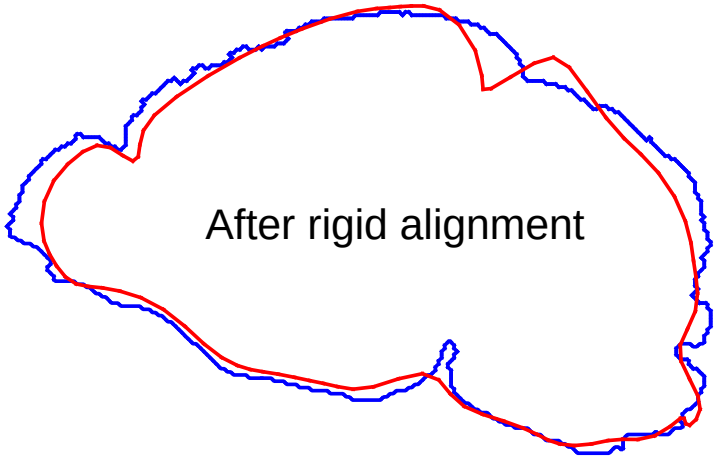

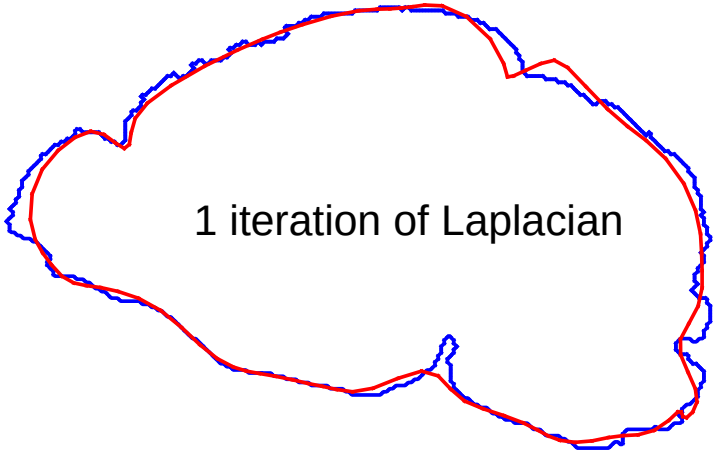

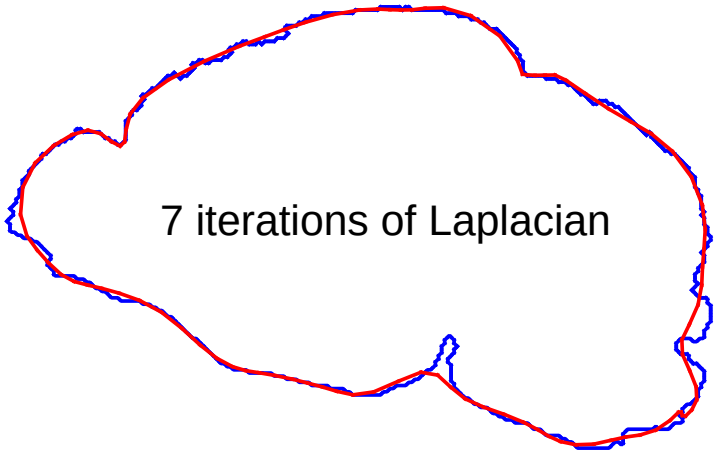

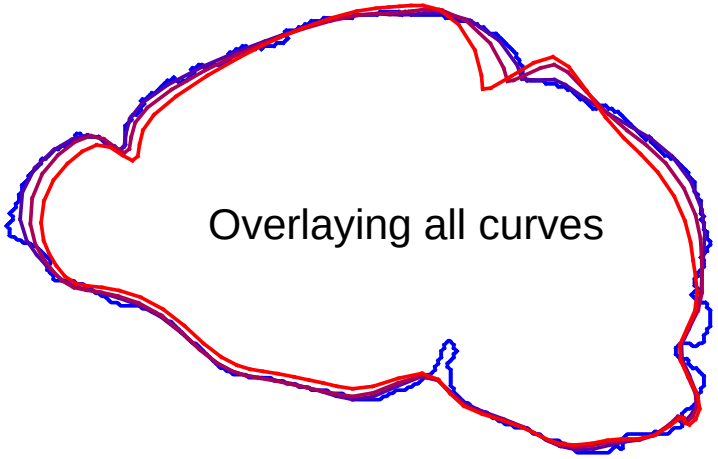

Result

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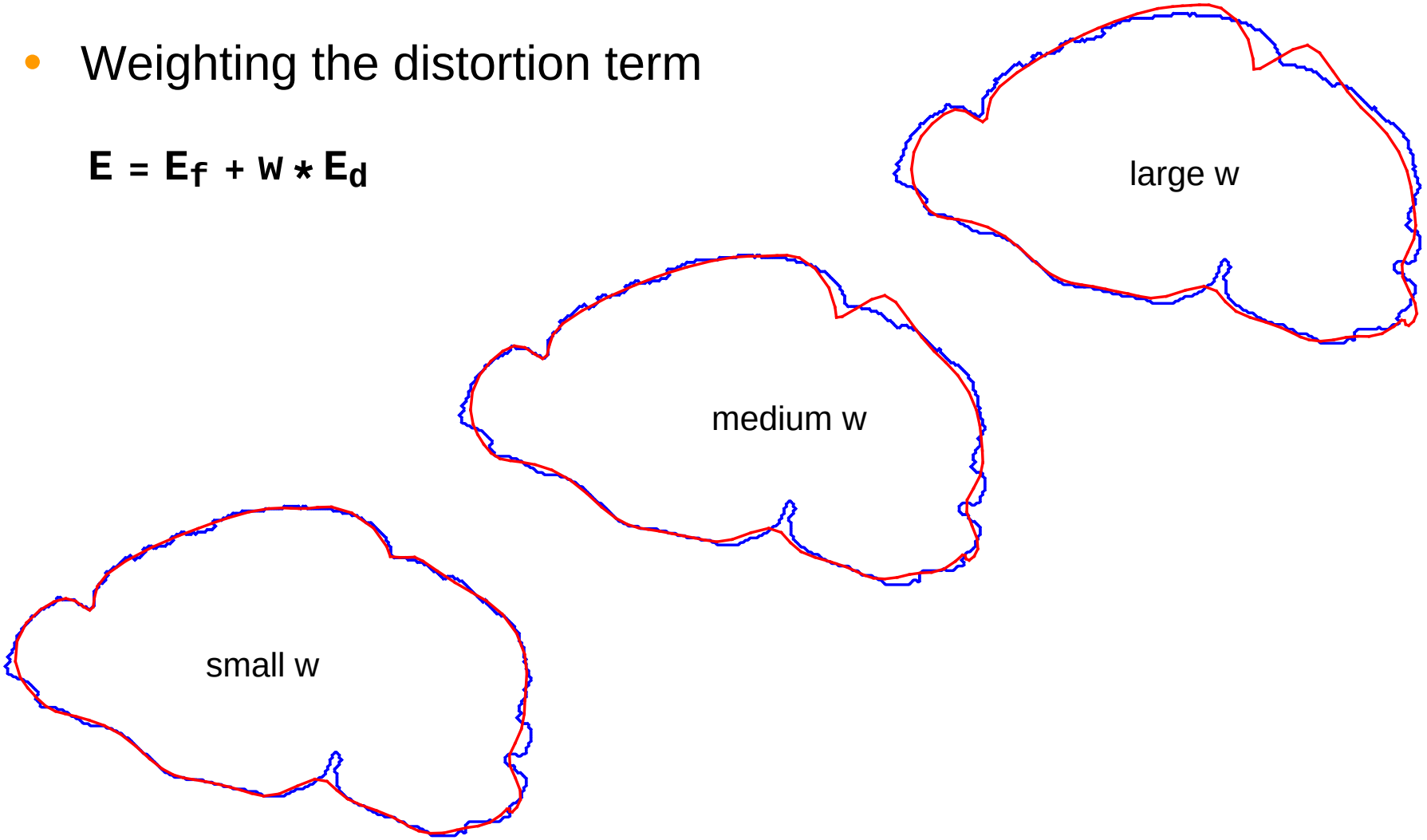
- Weighting the distortion term

$$E = E_f + w * E_d$$



Similarity Transforms (3D)

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- Elementary transformation matrices
  - To perform a sequence of transformations: take the product of these matrices

$$\begin{pmatrix} p_x' \\ p_y' \\ p_z' \\ 1 \end{pmatrix} = M \cdot \begin{pmatrix} p_x \\ p_y \\ p_z \\ 1 \end{pmatrix}$$

$$\mathbf{Trs}[v] = \begin{pmatrix} 1 & 0 & 0 & v_x \\ 0 & 1 & 0 & v_y \\ 0 & 0 & 1 & v_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Translation by vector v

$$\mathbf{Scl}[s] = \begin{pmatrix} s & 0 & 0 & 1 \\ 0 & s & 0 & 1 \\ 0 & 0 & s & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Scaling by scalar s

$$\mathbf{Rot}[X, \alpha] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \mathbf{Cos}[\alpha] & -\mathbf{Sin}[\alpha] & 0 \\ 0 & \mathbf{Sin}[\alpha] & \mathbf{Cos}[\alpha] & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Rotation by angle α around X axis

Similarity Transforms (3D)

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- General similarity transformations in 3D

$$T = \begin{pmatrix} s & -h_3 & h_2 & t_x \\ h_3 & s & -h_1 & t_y \\ -h_2 & h_1 & s & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

  - Approximates the product of a set of elementary matrices
    - Up to a small rotation angle
    - May introduce skewing for large rotations

Computing T_i (3D)

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- Assuming known deformation, by quadratic minimization:

$$\begin{pmatrix} s \\ h_1 \\ h_2 \\ h_3 \\ t_x \\ t_y \\ t_z \end{pmatrix} = (C^T C)^{-1} C^T \begin{pmatrix} p_{i_x}' \\ p_{i_y}' \\ p_{i_z}' \\ p_{i1_x}' \\ p_{i1_y}' \\ p_{i1_z}' \\ \vdots \end{pmatrix} \quad \text{where } C = \begin{pmatrix} p_{i_x} & 0 & p_{i_z} & -p_{i_y} & 1 & 0 & 0 \\ p_{i_y} & -p_{i_z} & 0 & p_{i_x} & 0 & 1 & 0 \\ p_{i_z} & p_{i_y} & -p_{i_x} & 0 & 0 & 0 & 1 \\ p_{i1_x} & 0 & p_{i1_z} & -p_{i1_y} & 1 & 0 & 0 \\ p_{i1_y} & -p_{i1_z} & 0 & p_{i1_x} & 0 & 1 & 0 \\ p_{i1_z} & p_{i1_y} & -p_{i1_x} & 0 & 0 & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

  - Linear expressions of the deformed points p_i'
    - C is a $3|N_i|+3$ by 7 matrix

Distortion Term (3D)

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- Constructing transformed Laplacian:

$$\mathbf{T}_i\,\delta_i = \mathbf{D}\begin{pmatrix} s \\ h_1 \\ h_2 \\ h_3 \\ t_x \\ t_y \\ t_z \end{pmatrix} = \mathbf{D}\left(\mathbf{C}^T\mathbf{C}\right)^{-1}\mathbf{C}^T\begin{pmatrix} p_{i_x}' \\ p_{i_y}' \\ p_{i_z}' \\ p_{i1_x}' \\ p_{i1_y}' \\ p_{i1_z}' \\ \vdots \end{pmatrix}$$

where $\mathbf{D} = \begin{pmatrix} \delta_{i_x} & 0 & \delta_{i_z} & -\delta_{i_y} & 1 & 0 & 0 \\ \delta_{i_y} & -\delta_{i_z} & 0 & \delta_{i_x} & 0 & 1 & 0 \\ \delta_{i_z} & \delta_{i_y} & -\delta_{i_x} & 0 & 0 & 0 & 1 \end{pmatrix}$