A Learning Market-Maker in the Glosten-Milgrom Model

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Abstract.

This paper develops a model of a learning market-maker by extending the Glosten-Milgrom model of dealer markets. The market-maker tracks the changing true value of a stock in settings with informed traders (with noisy signals) and liquidity traders, and sets bid and ask prices based on its estimate of the true value. We empirically evaluate the performance of the market-maker in markets with different parameter values to demonstrate the effectiveness of the algorithm, and then use the algorithm to derive properties of price processes in simulated markets. When the true value is governed by a jump process, there is a two regime behavior marked by significant heterogeneity of information and large spreads immediately following a price jump, which is quickly resolved by the market-maker, leading to a rapid return to homogeneity of information and small spreads. We also discuss the similarities and differences between our model and real stock market data in terms of distributional and time series properties of returns.

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1. Introduction

Glosten and Milgrom analyze the market-maker's decision problem in a stylized model with informed (insider) and uninformed (liquidity) traders [1]. This paper presents an algorithm for explicitly computing approximate solutions to the expected-value equations for setting prices in an extension of the Glosten-Milgrom model with probabilistic shocks to the underlying true price and noisy informed traders. We validate the algorithm by showing that it produces reasonable market-maker behavior across a range of simulations, and use the algorithm to study the time series and distributional properties of returns in our market model and compare them to real stock market data. We study the impact of different parameters on market properties, and in particular we show that there is a two regime behavior in which extreme heterogeneity of information immediately following a jump in the true value (characterized by high spreads and volatility) is quickly resolved and the market returns to a state of homogeneous information characterized by low spreads and volatility.

The particular model we investigate extends the Glosten-Milgrom model. Price-taking informed and uninformed traders interact through a price-setting market-maker. Informed traders receive a (potentially noisy) signal indicating the true underlying value of the stock and make buy and sell decisions based on the market-maker's quotes and the signal they receive. The true value receives periodic shocks drawn from a Gaussian distribution. Market-makers receive no information about the true value of the stock and must base their estimates solely on the order flow they observe. We simulate markets with both risk-neutral and simple risk-averse market-makers.

Glosten and Milgrom derive the market-maker's price setting equations under asymmetric information to be such that the bid quote is the expectation of the true value given that a sell order is received and the ask quote is the expectation of the true value given that a buy order is received. These expectations cannot be computed (except in "toy" instances) without maintaining a probability density estimate over the true value of the stock, especially when the true value itself may change. We introduce a nonparametric density estimation technique for maintaining a probability distribution over the true value that the market-maker can use to set prices. We also present a method to approximately solve the price setting equations in a realistic situation with dollars-and-cents quotes and prices. Market-makers using our algorithm in simulations can successfully achieve low spreads without incurring losses. Market-making agents are also often constrained by inventory control considerations brought about by risk aversion [2], so we study the effects of using an inventory control function that is added as an extra module to the market-making algorithm. The inventory control module greatly reduces the variance in market-maker profits.

Our simulations yield interesting market properties in different situations. Bid-ask spreads are higher in more volatile markets, market-makers increase the spread in response to uncertainty about the true price, the distribution of returns is leptokurtic, and the autocorrelation of raw returns decays rapidly.

Our approach to microstructure problems in dealer markets falls between the traditional theoretical models, such as those of Garman [3], Glosten and Milgrom [1] and Kyle [4] and the agent-based or artificial markets approach adopted by Darley [5] and Raberto *et al* [6] among others. We extend the theoretical model of Glosten and Milgrom into a more realistic setting which is nevertheless much simpler than most agent-based models. Our study of price properties in simulated markets is related to the burgeoning econophysics literature that studies statistical properties of price movements in real markets and their departures from the efficient market hypothesis [7, 8, 9, inter alia]. Much of the econophysics work attempts to work backward from real stock market data and model price impact functions and thus derive properties of price processes. The approach of this paper is complementary – it makes the traditional theoretical economic description richer and derives properties from the theory.

This paper is organized as follows. We explicitly define our market model and present the market-making algorithm in section 2 and experimentally evaluate the algorithm in simulated markets in section 3. In section 4, we present the distributional and time series properties of returns in our simulated markets, before summarizing in section 5.

2. The Market Model and Market-Making Algorithm

2.1. Market Model

The market we analyze is a discrete time dealer market with only one stock. The market-maker sets bid and ask prices (P_b and P_a respectively) at which it is willing to buy or sell one unit of the stock at each time period (when necessary we denote the bid and ask prices at time period i as P_b^i and P_a^i). If there are multiple market-makers, the market bid and ask prices are the maximum over each dealer's bid price and the minimum over each dealer's ask price. All transactions occur with the market-maker taking one side of the trade and a member of the trading crowd (henceforth a "trader") taking the other side.

The stock has a true underlying value (or fundamental value) V^i at time period i. All market makers are informed of V^0 at the beginning of a simulation, but do not receive any direct information about V after that‡. At time period i, a single trader is selected from the trading crowd and allowed to place either a (market) buy or (market) sell order for one unit of the stock. There are two types of traders in the market, uninformed traders and informed traders. An uninformed trader will place a buy or sell order for one unit with equal probability, or no order with some probability if selected to trade. An informed trader who is selected to trade knows V^i and will place a buy order if $V^i > P_a^i$, a sell order if $V^i < P_b^i$ and no order if $P_b^i \leq V^i \leq P_a^i$.

In addition to perfectly informed traders, we also allow for the presence of noisy informed traders. A noisy informed trader receives a signal of the true price $W^i = V^i + \tilde{\eta}(0, \sigma_W)$ where $\tilde{\eta}(0, \sigma_W)$ represents a sample from a normal distribution with mean 0 and variance σ_W^2. The noisy informed trader believes this is the true value of the stock, and places a buy order if $W^i > P_a^i$, a sell order if $W^i < P_b^i$ and no order if $P_b^i \leq W^i \leq P_a^i$.

The true underlying value of the stock evolves according to a jump process. At time $i+1$, with probability p, a jump in the true value occurs§. It is also possible to fix the periodicity of these jumps to model, for example, daily releases of information. When a jump occurs, the value changes according to the equation $V^{i+1} = V^i + \tilde{\omega}(0, \sigma)$ where $\tilde{\omega}(0, \sigma)$ represents a sample from a normal distribution with mean 0 and variance σ^2. Market-makers are informed of when a jump has occurred, but not of the size or direction of the jump.

‡ That is, the only signals a market-maker receives about the true value of the stock are through the buy and sell orders placed by the trading crowd.

§ p is typically small, of the order of 1 in 1000 in most of our simulations

This model of the evolution of the true value corresponds to the notion of the true value evolving as a result of occasional news items. The periods immediately following jumps are the periods in which informed traders can trade most profitably, because the information they have on the true value has not been disseminated to the market yet, and the market maker is not informed of changes in the true value and must estimate these through orders placed by the trading crowd. The market-maker will not update prices to the neighborhood of the new true value for some period of time immediately following a jump in the true value, and informed traders can exploit the information asymmetry.

2.2. The Market-Making Algorithm

The market-maker attempts to track the true value over time by maintaining a probability distribution over possible true values and updating the distribution when it receives signals from the orders that traders place. The true value and the market-maker's prices together induce a probability distribution on the orders that arrive in the market. The market-maker must maintain an online probabilistic estimate of the true value.

Glosten and Milgrom [1] analyze the setting of bid and ask prices so that the market maker enforces a zero profit condition. The zero profit condition corresponds to the Nash equilibrium in a setting with competitive market-makers. Glosten and Milgrom suggest that the market maker should set $P_b = E[V|\text{Sell}]$ and $P_a = E[V|\text{Buy}]$. Our market-making algorithm computes these expectations using the probability density function being estimated.

Various layers of complexity can be added on top of the basic algorithm. For example, minimum and maximum conditions can be imposed on the spread, and an inventory control mechanism could form another layer after the zero-profit condition prices are decided. We describe the density estimation technique in detail before addressing other possible factors that market-makers can take into account in deciding how to set prices. For simplicity of presentation, we neglect noisy informed traders in the initial derivation, and present the updated equations for taking them into account later.

2.2.1. Derivation of Bid and Ask Price Equations Let α be the proportion of informed traders in the trading crowd, and let η be the probability that an uninformed trader places a buy (or sell) order. Then the probability that an uninformed trader places no order is $1 - 2\eta$.

In order to estimate the expectation of the underlying value, it is necessary to compute the conditional probability that $V = x$ given that a particular type of order is received. Taking market sell orders as an example:

$$E[V|\text{Sell}] = \int_0^\infty x \Pr(V = x|\text{Sell})\, dx$$

To explicitly (approximately) compute these values, we discretize the X-axis into intervals, with each interval representing one cent. Then we get:

$$E[V|\text{Sell}] = \sum_{V_i=V_{\min}}^{V_i=V_{\max}} V_i \Pr(V = V_i|\text{Sell})$$

Applying Bayes' rule and simplifying:

$$E[V|\text{Sell}] = \sum_{V_i=V_{\min}}^{V_i=V_{\max}} \frac{V_i \Pr(\text{Sell}|V = V_i) \Pr(V = V_i)}{\Pr(\text{Sell})}$$

The *a priori* probability of a sell order (denoted by P_{Sell}) can be computed by taking advantage of the fact that informed traders will always sell if $V < P_b$ and never sell otherwise, while uninformed traders will sell with a constant probability:

$$P_{\text{Sell}} = \sum_{V_i=V_{\min}}^{V_i=V_{\max}} \Pr(\text{Sell}|V = V_i) \Pr(V = V_i)$$

$$= \sum_{V_i=V_{\min}}^{V_i=P_b-1} [(\alpha + (1-\alpha)\eta) \Pr(V = V_i)] + \sum_{V_i=P_b}^{V_i=V_{\max}} [((1-\alpha)\eta) \Pr(V = V_i)] \quad (1)$$

Since P_b is set by the market maker to $E[V|\text{Sell}]$:

$$P_b = \frac{1}{P_{\text{Sell}}} \sum_{V_i=V_{\min}}^{V_i=V_{\max}} V_i \Pr(\text{Sell}|V = V_i) \Pr(V = V_i)$$

Since $V_{\min} < P_b < V_{\max}$,

$$P_b = \frac{1}{P_{\text{Sell}}} \sum_{V_i=V_{\min}}^{V_i=P_b-1} V_i \Pr(\text{Sell}|V = V_i) \Pr(V = V_i) + \frac{1}{P_{\text{Sell}}} \sum_{V_i=P_b}^{V_i=V_{\max}} V_i \Pr(\text{Sell}|V = V_i) \Pr(V = V_i) \quad (2)$$

The term $\Pr(\text{Sell}|V = V_i)$ is constant within each sum, because of the influence of informed traders. An uninformed trader is equally likely to sell whatever the market maker's bid price. On the other hand, an informed trader will never sell if $V > P_b$. Therefore, $\Pr(\text{Sell}|V < P_b) = (1-\alpha)\eta + \alpha$ and $\Pr(\text{Sell}|V \geq P_b) = (1-\alpha)\eta$. The above equation reduces to:

$$P_b = \frac{1}{P_{\text{Sell}}} \left(\sum_{V_i=V_{\min}}^{V_i=P_b-1} ((1-\alpha)\eta + \alpha) V_i \Pr(V = V_i) + \sum_{V_i=P_b}^{V_i=V_{\max}} ((1-\alpha)\eta) V_i \Pr(V = V_i) \right) \quad (3)$$

Using a precisely parallel argument, we can derive the expression for the market-maker's ask price. First, we note that the prior probability of a buy order, P_{Buy} is:

$$P_{\text{Buy}} = \sum_{V_i=V_{\min}}^{V_i=P_a} [((1-\alpha)\eta) \Pr(V = V_i)] + \sum_{V_i=P_a+1}^{V_i=V_{\max}} [(\alpha + (1-\alpha)\eta) \Pr(V = V_i)] \quad (4)$$

Then P_a is the solution to the equation:

$$P_a = \frac{1}{P_{\text{Buy}}} \left(\sum_{V_i=V_{\min}}^{V_i=P_a} ((1-\alpha)\eta) V_i \Pr(V = V_i) + \sum_{V_i=P_a+1}^{V_i=V_{\max}} ((1-\alpha)\eta + \alpha) V_i \Pr(V = V_i) \right) \quad (5)$$

2.2.2. Accounting for Noisy Informed Traders An interesting feature of the probabilistic estimate of the true value is that the probability of buying or selling is the same conditional on V being smaller than or greater than a certain amount. For example, $\Pr(\text{Sell}|V = V_i, V_i \leq P_b)$ is a constant, independent of V. If we assume that all informed traders receive noisy signals, with the noise normally distributed with mean 0 and variance σ_W^2, and, as before, α is the proportion of informed traders in the trading crowd, then equation 2 still applies. Now the probabilities $\Pr(\text{Sell}|V = V_i)$ are no longer determined solely by whether $V_i < P_b$ or $V_i \geq P_b$. Instead, the new equations are:

$$\Pr(\text{Sell}|V = V_i, V_i \leq P_b) = (1-\alpha)\eta + \alpha \Pr(\tilde{\eta}(0, \sigma_W^2) < (P_b - V_i)) \quad (6)$$

$$\Pr(\text{Sell}|V = V_i, V_i > P_b) = (1-\alpha)\eta + \alpha \Pr(\tilde{\eta}(0, \sigma_W^2) > (V_i - P_b)) \quad (7)$$

The second term in the first equation reflects the probability that an informed trader would sell if the fundamental value were less than the market-maker's bid price. This will occur as long as $W = V + \tilde{\eta}(0, \sigma_W^2) < P_b$. The second term in the second equation reflects the same probability under the assumption that $V \geq P_b$.

We can compute the conditional probabilities for buy orders equivalently:

$$\Pr(\text{Buy}|V = V_i, V_i \leq P_a) = (1-\alpha)\eta + \alpha \Pr(\tilde{\eta}(0, \sigma_W^2) > (P_a - V_i)) \quad (8)$$

$$\Pr(\text{Buy}|V = V_i, V_i > P_a) = (1-\alpha)\eta + \alpha \Pr(\tilde{\eta}(0, \sigma_W^2) < (V_i - P_a)) \quad (9)$$

Now, we have the new buy and sell priors:

$$P_{\text{Sell}} = \sum_{V_i=V_{\min}}^{V_i=P_b-1} \left[\alpha \Pr(\tilde{\eta}(0, \sigma_W^2) < (P_b - V_i)) + (1-\alpha)\eta\right] \Pr(V = V_i) + \sum_{V_i=P_b}^{V_i=V_{\max}} \left[\alpha \Pr(\tilde{\eta}(0, \sigma_W^2) > (V_i - P_b)) + (1-\alpha)\eta\right] \Pr(V = V_i) \quad (10)$$

$$P_{\text{Buy}} = \sum_{V_i=V_{\min}}^{V_i=P_a} \left[\alpha \Pr(\tilde{\eta}(0, \sigma_W^2) > (P_a - V_i)) + (1-\alpha)\eta\right] \Pr(V = V_i) + \sum_{V_i=P_a+1}^{V_i=V_{\max}} \left[(\alpha \Pr(\tilde{\eta}(0, \sigma_W^2) < (V_i - P_a) + (1-\alpha)\eta)\right] \Pr(V = V_i) \quad (11)$$

We can substitute these conditional probabilities back into the fixed point equations and the density update rule used by the market-maker. Combining equations 2, 6 and

7, and using the sell prior from equation 10

$$P_b = \frac{1}{P_{\text{Sell}}} \sum_{V_i=V_{\min}}^{V_i=P_b} \left[((1-\alpha)\eta + \alpha \Pr(\tilde{\eta}(0,\sigma_W^2) < (P_b - V_i)))V_i \Pr(V = V_i)\right] +$$
$$\frac{1}{P_{\text{Sell}}} \sum_{V_i=P_b+1}^{V_i=V_{\max}} \left[((1-\alpha)\eta + \alpha \Pr(\tilde{\eta}(0,\sigma_W^2) > (V_i - P_b)))V_i \Pr(V = V_i)\right] \quad (12)$$

Similarly, for the ask price, using the buy prior from equation 11:

$$P_a = \frac{1}{P_{\text{Buy}}} \sum_{V_i=V_{\min}}^{V_i=P_a} \left[((1-\alpha)\eta + \alpha \Pr(\tilde{\eta}(0,\sigma_W^2) > (P_a - V_i)))V_i \Pr(V = V_i)\right] +$$
$$\frac{1}{P_{\text{Buy}}} \sum_{V_i=P_a+1}^{V_i=V_{\max}} \left[((1-\alpha)\eta + \alpha \Pr(\tilde{\eta}(0,\sigma_W^2) < (V_i - P_a)))V_i \Pr(V = V_i)\right] \quad (13)$$

2.2.3. Approximately Solving the Equations A number of problems arise with the analytical solution of these discrete equations for setting the bid and ask prices. Most importantly, we have not yet specified the probability distribution for priors on V, and any reasonably complex solution leads to a form that makes analytical solution infeasible. Secondly, the values of $V_{\min}$ and $V_{\max}$ are undetermined. And finally, actual solution of these fixed point equations must be approximated in discrete spaces. We solve each of these problems in turn to construct an approximate solution to the problem.

We assume that the market-making agent is aware of the true value at time 0, and from then onwards the true value infrequently receives random shocks (or jumps) drawn from a normal distribution (the variance of which is known to the agent). Our market-maker constructs a vector of prior probabilities on various possible values of V as follows.

If the initial true value is V_0 (when rounded to an integral value in cents), then the agent constructs a vector going from $V_0 - 4\sigma$ to $V_0 + 4\sigma - 1$ to contain the prior value probabilities. The probability that $V = V_0 - 4\sigma + i$ is given by the ith value in this vector‖. The vector is initialized by setting the ith value in the vector to $\int_{-4\sigma+i}^{-4\sigma+i+1} \mathcal{N}(0,\sigma)\,dx$ where $\mathcal{N}$ is the normal density function in x with specified mean and variance. The vector is maintained in a normalized state at all times so that the entire probability mass for V lies within it.

The fixed point equations 12 and 13 are approximately solved by using the result from Glosten and Milgrom that $P_b \le E[V] \le P_a$ and then, to find the bid price, for example, cycling from E[V] downwards until the difference between the left and right hand sides of the equation stops decreasing. The fixed point real-valued solution must then be closest to the integral value at which the distance between the two sides of the equation is minimized.

‖ The true value can be a real number, but for all practical purposes it ends up getting truncated to the floor of that number.

2.2.4. Updating the Density Estimate The market-maker receives probabilistic signals about the true value. With perfectly informed traders, each signal says that with a certain probability, the true value is lower (higher) than the bid (ask) price. With noisy informed traders, the signal differentiates between different possible true values depending on the market-maker's bid and ask quotes. Each time that the market-maker receives a signal about the true value by receiving a market buy or sell order, it updates the posterior on the value of V by scaling the distributions based on the type of order. The Bayesian updates are easily derived. For example, for $V_i \leq P_a$ and market buy orders:

$$\Pr(V = V_i|\text{Buy}) = \frac{\Pr(\text{Buy}|V = V_i)\Pr(V = V_i)}{\Pr(\text{Buy})}$$

The prior probability $V = V_i$ is known from the density estimate, the prior probability of a buy order is known from equation 11, and $\Pr(\text{Buy}|V = V_i, V_i \leq P_a)$ can be computed from equation 8. We can compute the posterior similarly for each of the cases. One case that is instructive to look at since we have not derived it above is the case when no order is received.

$$\Pr(V = V_i|\text{No order}) = \frac{\Pr(\text{No order}|V = V_i)\Pr(V = V_i)}{\Pr(\text{No order})}$$

Now,

$$\Pr(\text{No order}|V = V_i, V_i < P_b) = (1-\alpha)(1-2\eta) + \alpha\Pr(\tilde{\eta}(0,\sigma_W^2) > (P_b - V_i))$$

$$\Pr(\text{No order}|V = V_i, P_b \leq V_i \leq P_a) = (1-\alpha)(1-2\eta) + \alpha\left[\Pr(P_b - V_i < \tilde{\eta}(0,\sigma_W^2)) + \Pr(V_i - P_a < \tilde{\eta}(0,\sigma_W^2))\right]$$

$$\Pr(\text{No order}|V = V_i, V_i > P_a) = (1-\alpha)(1-2\eta) + \alpha\Pr(V_i - P_a < \tilde{\eta}(0,\sigma_W^2))$$

which allows us to compute the prior as well as all the terms in the numerator.

In the case of perfectly informed traders, the signal only specifies that the true value is higher or lower than some price, and not how much higher or lower. In that case, the update equations are as follows. If a market buy order is received, this is a signal that with probability $(1-\alpha)\eta + \alpha, V > P_a$. Similarly, if a market sell order is received, the signal indicates that with probability $(1-\alpha)\eta + \alpha, V < P_b$.

In the former case, all probabilities for $V = V_i, V_i > P_a$ are multiplied by $(1-\alpha)\eta+\alpha$, while all the other discrete probabilities are multiplied by $1-\alpha-(1-\alpha)\eta$. Similarly, when a sell order is received, all probabilities for $V = V_i, V_i < P_b$ are multiplied by $(1-\alpha)\eta+\alpha$, and all the remaining discrete probabilities are multiplied by $1-\alpha-(1-\alpha)\eta$ before renormalizing.

These updates lead to less smooth density estimates than the updates for noisy informed traders, as can be seen from figure 1, which shows the density functions 0, 3 and 6 steps after a jump in the underlying value of the stock. The update equations that consider noisy informed traders smoothly transform the probability distribution around the last transaction price by a mixture of a Gaussian and a uniform density, whereas the update equations for perfectly informed traders discretely shift all probabilities to one side of the transaction price in one direction and on the other side of the transaction

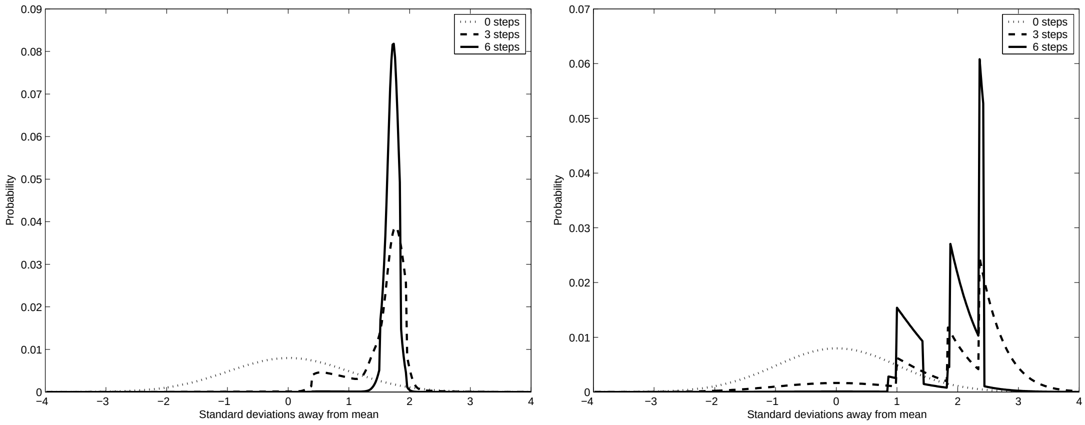

Figure 1. The evolution of the market-maker's probability density estimate with noisy informed traders (left) and perfectly informed traders (right)

price in the other direction. The estimates for perfectly informed traders are more susceptible to noise, as they do not restrict most of the mass of the probability density function to as small an area as the estimates for noisy informed traders.

3. Experimental Evaluation

3.1. Experimental Framework

Unless specified otherwise, it can be assumed that all simulations take place in a market populated by noisy informed traders and uninformed traders. The noisy informed traders receive a noisy signal of the true value of the stock with the noise term being drawn from a Gaussian distribution with mean 0 and standard deviation 5 cents. The standard deviation of the jump process for the stock is 50 cents, and the probability of a jump occurring at any time step is 0.001. The probability of an uninformed buy or sell order is 0.5. The market-maker is informed of when a jump occurs, but not of the size or direction of the jump. The market-maker may use an inventory control function (defined below) and can increase the spread by lowering the bid price and raising the ask price by a fixed amount (this is done to ensure profitability and is also explained below). We report average results from 200 simulations, each lasting 50,000 time steps.

3.2. Prices Near a Jump

Figure 2 shows that the market-maker successfully tracks the true value over the course of an entire simulation. These results are from a simulation with half the traders being perfectly informed and the other half uninformed. The bid-ask spread reflects the market-maker's uncertainty about the true value — for example, it is much higher immediately after the true value has jumped.

Figure 2 also demonstrates that the asymmetry of information immediately following a price jump gets resolved very quickly. To investigate this further, we tracked

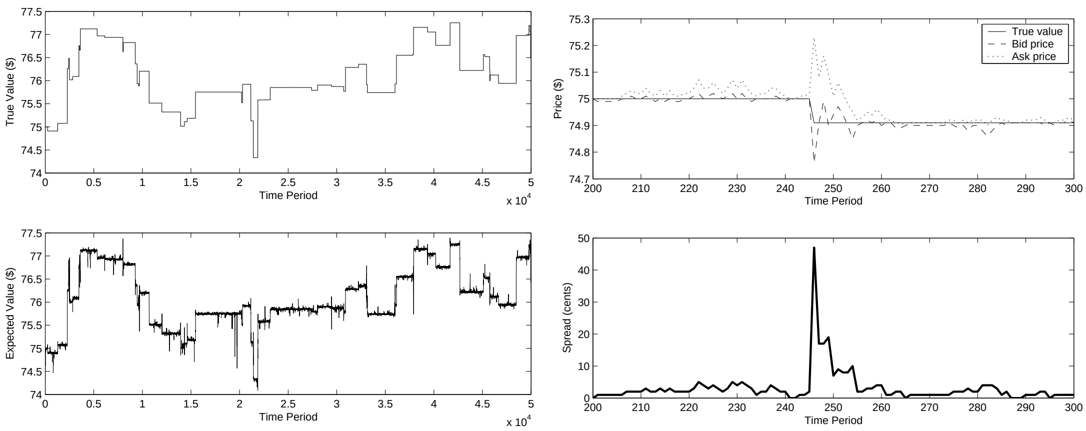

Figure 2. The market-maker's tracking of the true price over the course of the simulation (left) and immediately before and after a price jump (right)

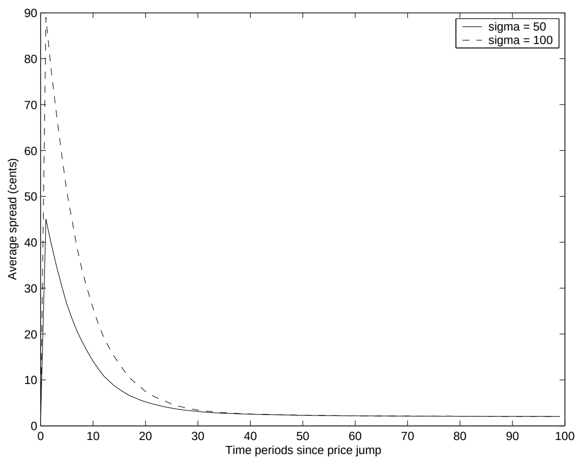

Figure 3. Average spread following a price jump for two different values of the standard deviation of the jump process

the average spread immediately following a price jump in a similar market environment (except with noisy informed traders instead of perfectly informed ones). The results of this experiment are shown in figure 3. It is clear that the informational asymmetry gets resolved very quickly (within thirty trades) independently of the standard deviation of the jump process.

3.3. Profit Motive

The zero-profit condition of Glosten and Milgrom is expected from game theoretic considerations when multiple competitive dealers are making markets in the same stock. However, since our method is an approximation scheme, the zero profit method is unlikely to truly be zero-profit. Further, the market-maker is not always in a perfectly competitive scenario where it needs to restrict the spread as much as possible.

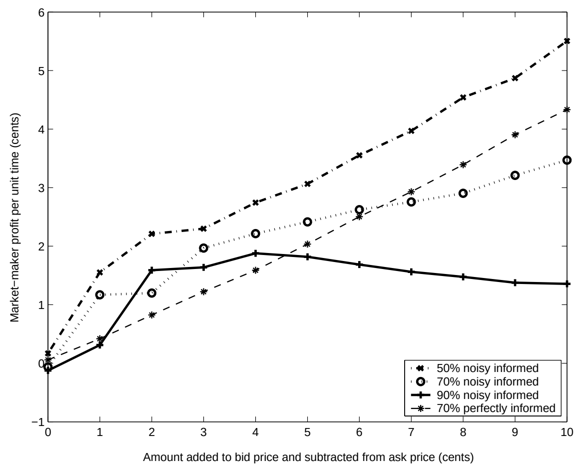

Figure 4. Market-maker profits as a function of increasing the spread

The simplest solution to the problem of making profit is to increase the spread by pushing the bid and ask prices apart after the zero-profit bid and ask prices have been computed using the density estimate obtained by the market-making algorithm. Figure 4 shows the profit obtained by a single monopolistic market-maker in markets with different percentages of informed traders. The numbers on the X axis show the amount (in cents) that is subtracted from (added to) the zero-profit bid (ask) price in order to push the spread apart (we will refer to this number as the shift factor).

With lower spreads, most of the market-maker's profits come from the noise factor of the informed traders, whereas with a higher spread, most of the market-maker's profits come from the trades of uninformed traders. Different percentages of informed traders lead to differently shaped profit curves. For example, there is a sharper jump in the transition from a shift factor of 0 to a shift factor of 1 with fewer noisy informed traders (50% or 70%) whereas with 90% noisy informed traders the sharper jump is in going from a shift factor of 1 to a shift factor of 2. With only 50% of the traders being informed, the market-maker's profit keeps increasing with the size of the spread.

However, increasing the spread beyond a point is counterproductive if there are enough noisy informed traders in the markets, because then the market-maker's prices are far enough away from the true value that even the noise factor cannot influence the informed traders to make trades. With 90% of the traders being informed, a global maximum (at least for reasonable spreads) is attained with a low spread. This is where the tradeoff between not increasing the spread too much in order to profit from the noise in the informed traders' signals and increasing the spread more to profit more from uninformed traders is optimized. On the opposite end of the spectrum, the market-maker's profits increase smoothly and uniformly with the spread when there are only perfectly informed traders in the market in addition to the uninformed traders, since all the market-maker's profits are from the uninformed traders.

It is important to note that market-makers can make reasonable profits with low average spreads. For a market with 70% of the trading crowd consisting of noisy informed traders and the remaining 30% consisting of uninformed traders, our algorithm, using a shift factor of 1, achieves an average profit of 1.17 cents per time period with an average spread of 2.28 cents. Using a shift factor of 0, the average profit is −0.06 cents with an average spread of 0.35 cents. These parameter settings in this environment yield a market-maker that is close to a Nash equilibrium player, and it is unlikely that any algorithm would be able to outperform this one in direct competition in such an environment given the low spread. An interesting avenue to explore is the possibility of adaptively changing the shift factor depending on the level of competition in the market. Clearly, in a monopolistic setting, a market-maker is better off using a high shift factor, whereas in a competitive setting it is likely to be more successful using a smaller one. An algorithm for changing the shift factor based on the history of other market-makers' quotes would be useful.

3.4. Inventory Control

Stoll analyzes dealer costs in conducting transactions and divides them into three categories [10]. These three categories are portfolio risk, transaction costs and the cost of asymmetric information. In the model we have presented so far, following Glosten and Milgrom [1], we have assumed that transactions have zero execution cost and developed a pricing mechanism that explicitly attempts to set the spread to account for the cost of asymmetric information. A realistic model for market-making necessitates taking portfolio risk into account and controlling inventory in setting bid and ask prices. In the absence of consideration of trade size and failure conditions, portfolio risk should affect the placement of the bid and ask prices, but not the size of the spread¶ [2, 10, 11]. If the market-maker has a long position in the stock, minimizing portfolio risk is achieved by lowering both bid and ask prices, and if the market-maker has a short position, inventory is controlled by raising both bid and ask prices.

Inventory control can be incorporated into the architecture of our market-making algorithm by using it as an adjustment parameter applied after bid and ask prices have been determined by equations 12 and 13. A simple inventory control technique we investigate here is to raise or lower the bid and ask prices by a linear function of the inventory holdings of the market-maker. The amount added to the bid and ask prices is $-\gamma I$ where I is the amount of inventory held by the market-maker (negative for short positions) and γ is a risk-aversion coefficient.

Table 1 shows statistics indicating the effectiveness of the inventory control module for minimizing market-maker risk and the effects of using different values of γ. The figures use the absolute value of the difference between last true value and initial true value as a proxy for market volatility. 200 simulations were run for each experiment, and 70% of the traders were noisy informed traders, while the rest were uninformed.

¶ One would expect spread to increase with the trade size. The size of the spread is, of course, affected by the adverse selection arising due to the presence of informed traders.

γ	0	0.1	1
Average (absolute) inventory holdings	1387.2	9.74	1.66
Profit (cents per period)	1.169	0.757	0.434
Standard Deviation of profit	9.3813	0.0742	0.0178

Table 1. Average absolute value of MM inventory at the end of a simulation, average profit achieved and standard deviation of per-simulation profit for market-makers with different levels of inventory control

Shift	σ	p	Spread	Profit
0	50	.001	0.3479	-0.0701
0	50	.005	1.6189	-0.1295
0	100	.001	0.6422	-0.0694
0	100	.005	3.0657	-0.2412
1	50	.001	2.3111	0.7738
1	50	.005	3.5503	0.6373
1	100	.001	2.6142	0.7629
1	100	.005	4.9979	0.6340

Table 2. Market-maker average spreads (in cents) and profits (in cents per time period) as a function of the shift (amount added to ask price and subtracted from bid price), standard deviation of the jump process (σ) and the probability of a jump occurring at any point in time (p)

The market-maker used a shift factor of 1 for increasing / decreasing the ask / bid prices respectively.

3.5. The Effects of Volatility

Volatility of the underlying true value process is affected by two parameters. One is the standard deviation of the jump process, which affects the variability in the amount of each jump. The other is the probability with which a jump occurs. Table 2 shows the result of changing the standard deviation σ of the jump process and the probability p of a jump occurring at any point in time. As expected, the spread increases with increased volatility, both in terms of σ and p. The precise dependence of the expected profit and the average spread on the values of σ and p are interesting. For example, increasing p for $\sigma = 100$ has a more significant percentage impact on the spread than the same increase when $\sigma = 50$. This is probably because the mean reflects the relative importance of the symmetric and asymmetric information regimes, which is affected by p.

Case	Profit	Loss of expectation	Average spread
Known	0.7693	0.7546	2.3263
Unknown	−0.6633	4.5616	4.3708

Table 3. Average profit (in cents per time period), loss of expectation and average spread (cents) with jumps known and unknown

3.6. Accounting for Jumps

The great advantage of our algorithm for density estimation and price setting is that it quickly restricts most of the probability mass to a relatively small region of values/prices, which allows the market-maker to quote a small spread and still break even or make profit. The other side of this equation is that once the probability mass is concentrated in one area, the probability density function on other points in the space becomes small. In some cases, it is not possible to seamlessly update the estimate through the same process if a price jump occurs. Another problem is that a sequence of jumps could lead to the value leaving the $[-4\sigma, 4\sigma]$ window used by the density estimation technique.

In the discussion above, we have assumed that the market-maker is explicitly informed of when a price jump has occurred, although not of the size or direction of the jump. The problem can be solved by recentering the distribution around the current expected value and reinitializing in the same way in which the prior distribution on the value is initially set up. The "unknown jump" case is more complicated. An interesting avenue for future work, especially if trade sizes are incorporated into the model, is to devise a formal mathematical method for deciding when to recenter the distribution. An example of such a method would be to learn a classifier that is good at predicting when a price jump has occurred. Perhaps there are particular types of trades that commonly occur following price jumps, especially when limit orders and differing trade sizes are permitted. Sequences of such trades may form patterns that predict the occurrence of jumps in the underlying value.

An example of a very simple rule that demonstrates the feasibility of such an idea is to recenter based on some notion of order imbalance. Such a rule could recenter when there have been k more buy orders than sell orders (or vice versa) in the last n time steps. Table 3 shows the results obtained using $n = 10$ and $k = 5$ values with the market-maker increasing (decreasing) the ask (bid) price by 1 cent beyond the zero profit case, and using linear inventory control with $\gamma = 0.1$. The loss of the expectation is defined as the average of the absolute value of the difference between the true value and the market-maker's expectation of the true value at each point in time. While this rule makes a loss, the spread is reasonable and the expectation is not too far away on average from the true value. This demonstrates that it is reasonable to assume that there are endogenous ways for market-makers to know when jumps have occurred.

4. Time Series and Distributional Properties of Returns

We can utilize the market and price-setting models developed so far in order to derive price properties in the simulated market and compare these properties to what is seen in real markets by analyzing return data for ten stocks from the TAQ database. Obviously our model's simplicity means that it will not capture many of the features of real data. This discussion is intended to highlight where our model agrees and disagrees with real data and to hypothesize why these differences occur and whether they can be accommodated by additions to the model.

We use standardized log returns for all discussion in this section, using fifty discrete time periods as the length of time for simulation returns and five minutes for stock returns. The simulation data we report is averaged over 100 runs of 50,000 time steps each with 70% informed traders and the market-maker using inventory control with $\gamma = 1$ and increasing the ask price and decreasing the bid price by one cent beyond the zero-profit computation in order to ensure profitability. The probability of a jump in the true value at any time, $p = 0.005$ and the standard deviation of the jump process $\sigma = 50$ (cents). The stock data from the TAQ database is for ten randomly selected component stocks of the S&P 500 index for the year 2002[+].

Liu *et al* present a detailed analysis of the time series properties of returns in a real equity market (they focus on the S&P 500 and component stocks) [12]. Their major findings are that return distributions are leptokurtic and fat-tailed, with power-law decay in the tails, volatility clustering occurs and the autocorrelation of absolute values of returns is persistent over large time scales (again with power-law decay), as opposed to the autocorrelation of raw returns, which disappears rapidly[*]. The recent econophysics literature has seen a growing debate about the origin of power laws in such data, for example, the theory of Gabaix *et al* [7, 13] and the alternative analysis of Farmer and Lillo [8].

Our simulation results show rapid decay of autocorrelation of raw returns (the coefficient is already at noise levels at lag 1). Bouchaud *et al* [9] discuss how prices are a random walk because of a critical balance between liquidity takers who place market orders and create temporal correlations in the sign of trades because they do not wish to place huge orders that move the market immediately, and liquidity providers who attempt to mean-revert the price. In our model, all the traders except the market maker are liquidity takers, and they have an even harsher restriction on the trade size they can place. We explicitly model the price-setting process of the liquidity provider, and our results show that the autocorrelation of raw returns decays rapidly and arbitrage opportunities do not arise.

[+] The symbols for the ten stocks are CA, UNP, AMAT, GENZ, GLK, TNB, PMTC RX, UIS and VC. Of these the first four are considered large cap (with market capitalizations in excess of 6 billion dollars) and the other six are small cap.

[*] Liu *et al* are not the first to discover these properties of financial time series. However, they summarize much of the work in an appropriate fashion and provide detailed references, and they present novel results on the power law distribution of volatility correlation.

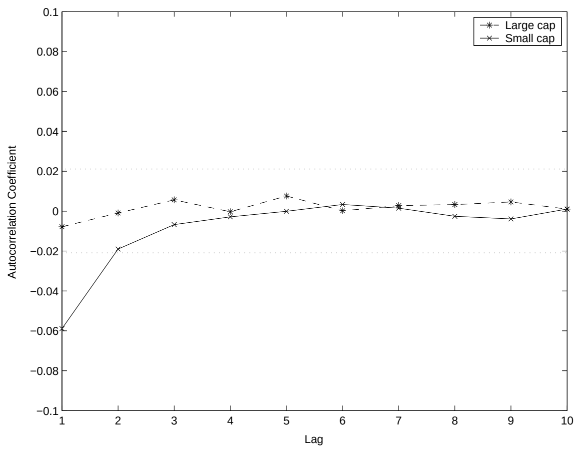

Figure 5. Autocorrelation of raw returns for small and large cap stocks (the dotted lines represent noise levels computed as $\pm 3/\sqrt{N}$ where N is the length of the time series

Looking at the real data, there is a negative serial correlation of raw returns at one lag for the small cap stocks (figure 5). This may be because of less trading in these stocks. In our model, we do not get this spike if we look at price changes over fifty periods, but if we look at them over fewer discrete time periods (say one or two) instead of fifty, we can see a statistically significant negative autocorrelation at one lag as well.

In terms of absolute returns, the real data shows a pronounced daily trend, with the autocorrelation coefficient spiking at the lag corresponding to one day (figure 6). This one day periodicity probably corresponds to opening and closing procedures (which also cause the spread to widen). We can replicate part of this phenomenon in our model by fixing the periodicity of shocks to the true value. This part corresponds to our intuition of major information shocks coming at the beginning of trading days, and induces a concept of the beginning of the trading day among agents in the market. However, it is hard to model the fact that the autocorrelation coefficient is higher for lags corresponding to, say three-quarters of a day in this model. This is because the agents in our model don't have a notion of the market closing, which may be what drives up the coefficient for these lags in real data. Perhaps a model in which the market-maker is sure that a price jump has occurred at the beginning of trading days, but also assumes the possibility of unknown jumps later in the day could explain these facts.

Simulations using our model yield return distributions with some similarities to stock market data, as can be seen from figure 7. The distribution of returns in our simulations is leptokurtic, although it does not decay with a power law tail, suggesting that our model needs further extensions before contributing to the debate on their origin. A huge proportion of the returns are very small, and virtually all of these occur in the symmetric information regime, and there are very few large returns, most of which occur in the asymmetric information regime immediately following a price jump.

The sample kurtosis for the simulation return data is 49.49 (by way of comparison,

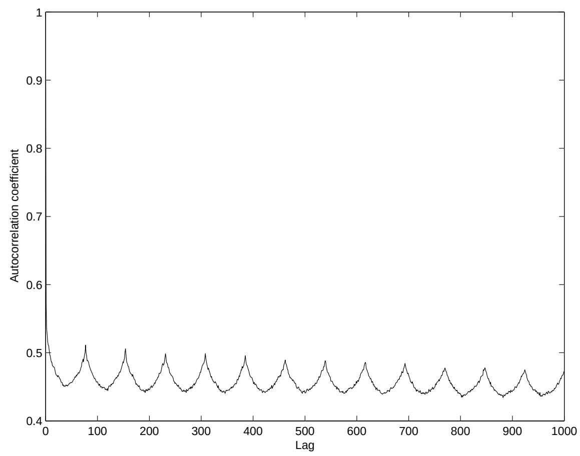

Figure 6. Autocorrelation of absolute returns for stock data

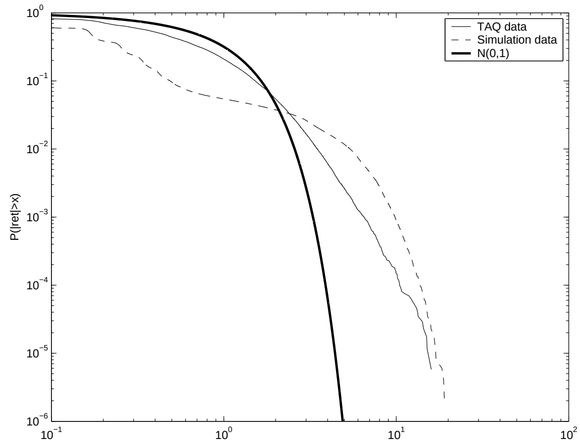

Figure 7. Distribution of absolute returns for simulation data and stock data, along with the cumulative distribution of the absolute value of a random variable drawn from a standard normal distribution

the sample kurtosis for the large cap stocks is 19.49 and that for the small cap stocks is 13.18). The exact shape of the distribution is affected by parameters like artificial inflation of the spread and inventory control. If the market-maker were to dynamically change the spread during the course of a simulation based on factors like competition or the need to maintain market quality, perhaps that would yield power law tails.

5. Summary

This paper extends the Glosten-Milgrom model of dealer markets by describing an algorithm for maintaining a probability density estimate of the true value of a stock in a dynamic market with regular shocks to the value and using this estimate to explicitly set prices in a somewhat realistic framework. We explicitly incorporate noise into the specification of informed trading, allowing for a rich range of market behavior. A careful empirical evaluation of characteristics of the market-making algorithm in simulation yields helpful insights for the problem of designing a market-making agent. Further, this framework allows us to develop an agent-based model of a dealer market and study time series, distributional and other properties of prices, and interesting interactions between different parameters.

There are two regimes in the simulated markets. Immediately following a price jump, information is very heterogeneous, spreads are high, and the market is volatile. This informational asymmetry gets resolved rapidly, and the market settles into a regime of homogeneous information with small spreads and low volatility. Analyzing time series and distributional properties of returns in our model shows some similarities and some differences from real data. The differences, in particular, could serve as a starting point for further extensions of this model to computationally study the effects of information and explicit modeling of the true value process in price formation.

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