Fair Division in Theory and Practice

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Lecture 8: Power Indices
Weighted voting

In 1958, six Western European countries formed the European Economic Community (EEC). Here was the vote breakdown:

Germany, France, Italy: 4 votes each
Belgium, Netherlands: 2 votes each
Luxembourg: 1 vote

The decision rule was 12 of 17 votes

Is there a dummy voter?
(A voter whose vote can never affect any outcome?)
Quantifying “power”

Whether or not a voter is a dummy is not monotone in the quota used or that voter’s voting weight — it depends on the particular set of winning coalitions that a weighted voting system generates.

Today we will discuss several quantitative measures of political power that take these configurations into account (power indices).

Each measure tries to quantify a voter’s control over outcomes.
What does equalizing power mean?

Power indices can help us determine whether or not a voter’s ability to affect outcomes corresponds in a logical way to his voting weight.

How is “one person, one vote” best implemented in a weighted voting system? Equalizing vote share per person, or equalizing voting power per person?
Shapley-Shubik power index preliminaries (combinatorics)

Suppose we have $n$ voters. How many different ways can we order them?

- $n = 2$: $(12), (21)$, so 2 different ways
- $n = 3$: $(123), (132), (213), (231), (312), (321)$, so 6
  - Look at $n = 2$ case; For each ordering there are three places we can place voter 3 (start, middle, end) — there are 3 times 2 possible orderings
- For $n = 4$, look at the $n = 3$ orderings; for each one, there are 4 different places we can place voter 4: $(\ast 1 \ast 2 \ast 3 \ast)$
Ordering voters

- Since there are 6 different orderings when $n = 3$, and 4 different places to put Voter 4 in each of those orderings, there are $4 \times 6$ orderings of 4 voters, or 24 orderings.
  - This is the same as $4 \times 3 \times 2 \times 1$

- When $n = 5$, there are 5 places to put Voter 5 in each of those orderings, so there are $5 \times 24$ orderings
  - This is $5 \times 4 \times 3 \times 2 \times 1$

- For $n$ people, there $n \times (n - 1) \times (n - 2) \times \ldots \times 2 \times 1$ orderings of those people

- This is written $n!$
Pivotal players

Suppose we have 7 voters. There are 7! combinations of those voters. Take a particular one:

\[(3 \ 5 \ 1 \ 6 \ 7 \ 4 \ 2)\]

- One player will be **pivotal** for this particular ordering
- Picture a larger and larger coalition being formed as we move left to right
- Adding one person to the coalition we are forming will turn it from losing into a (minimal) winning coalition — that person is the pivotal voter for this ordering
Pivotal voters

(3 5 1 6 7 4 2)

Suppose each voter has one vote except for person 4, who has three votes.

Suppose 5 votes are required for passage.

Who is the pivotal voter for this ordering?

Who is pivotal for the ordering (3 4 1 6 7 5 2)?
Shapley-Shubik Indices

A person’s Shapley-Shubik index is the fraction of orderings for which he is the pivotal voter.

With $n$ voters, Voter $i$’s Shapley-Shubik index is:

$$\frac{\text{Number of orderings for which Voter } i \text{ is pivotal}}{n!}$$

Note that every person’s index is between 0 and 1, and that the indices sum to 1.
Question

Suppose there are three voters with the following voting weights: (50, 49, 1), and that the voting quota is 51.

What is each person’s Shapley-Shubik index (SSI)?
Answer

Suppose there are three voters with the following voting weights: (50, 49, 1), and that the voting quota is 51

What is each person’s Shapley-Shubik index?

- (123): 2
- (132): 3
- (213): 1
- (231): 1
- (312): 1
- (321): 1

The voters’ indices are \( (\frac{4}{6}, \frac{1}{6}, \frac{1}{6}) \)
What are the players’ power indices when they have voting weights (2, 2, 1) and 3 votes are required for passage?
A real-world example

European Economic Community 1958 (12 of 17 votes, $\approx 70\%$)
Germany, France, Italy: 4 votes each
Belgium, Netherlands: 2 votes each
Luxembourg: 1 vote

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<th>Country</th>
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<th>Voting weight</th>
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“Paradox of new members”

Suppose new members are added to the system and given votes, but that the total percentage of votes required for passage stays the same.

We would expect the “power” of the original voters to become diluted or (stay the same) because their voting weights have decreased, but this is not the case.
Revised EEC voting rule

1973 EEC (41 of 58 votes needed for passage, ≈ 70%)

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Luxembourg’s power would have increased even if it had been left with just one vote.
Banzhaf Indices

Introduced by attorney John Banzhaf in connection with a lawsuit involving Nassau County board of supervisors (1965)

We calculate this index by looking at all decisive coalitions. Look at each member of each coalition. If removing that member makes the coalition no longer decisive, then give him a point. That voter is a critical voter for that coalition.

When finished, divide by the total number of points, and you have calculated each person’s Banzhaf power index
In-class question

Previous example of three voters with the following voting weights: (50, 49, 1), and voting quota of 51

The decisive coalitions are:

(13), (12), (123)

In each coalition, who is a critical voter? (There may be several or none per coalition)
Answer

Previous example of three voters with the following voting weights: 
(50, 49, 1), and voting quota of 51.

The decisive coalitions are:

\[(13), (12), (123)\]

- 1 and 3 are critical for (13)
- 1 and 2 are critical for (12)
- Only 1 is critical for (123)

The voters’ Banzhaf indices are \((\frac{3}{5}, \frac{1}{5}, \frac{1}{5})\)

Recall that the voters’ SSI’s were \((\frac{4}{6}, \frac{1}{6}, \frac{1}{6})\)
In-class question

Consider a weighted “quota rule” (or \( q \)-rule), with quota \( q = 6 \) and \( w_4 = 4, w_3 = 3, w_2 = 2 \) and \( w_1 = 1 \). The winning coalitions are:

\[
(4, 3), (4, 2), (4, 3, 2), (4, 3, 2, 1), (4, 3, 1), (4, 2, 1), (3, 2, 1)
\]

Find the Banzhaf indices of the voters – remember, find the “critical voters” for each coalition, give them each a point, and sum up these points.
Answer

Consider a weighted voting rule, with $q = 6$ and $w_4 = 4, w_3 = 3, w_2 = 2$ and $w_1 = 1$. The winning coalitions are:

$$(4, 3), (4, 2), (4, 3, 2), (4, 3, 2, 1), (4, 3, 1), (4, 2, 1), (3, 2, 1)$$

The critical voters in each coalition are:

$$(4^*, 3^*), (4^*, 2^*), (4^*, 3, 2), (4, 3, 2, 1), (4^*, 3^*, 1), (4^*, 2^*, 1), (3^*, 2^*, 1^*)$$

There are 12 critical votes, and the breakdown is:
4 (5), 3 (3), 2 (3), 1 (1), and so the Banzhaf indices are:

$5/12, 3/12, 3/12, 1/12$.

Thus, although 2 and 3 have different numbers of votes, they have the same voting power, given a (strong) assumption that any one of these situations in which a critical voter arises is equally likely (see Gelman and Katz, 2002)
Weighted voting in the U.S.

- Prior to 1960’s several states had county-level Boards of Supervisors composed of elected supervisors from constituent cities or towns
  - These towns could differ greatly in population
- Following “one man, one vote” mandate (Reynolds v. Sims, 1964), most such states changed these legislative bodies
- 24 of NY’s 57 counties did not, and instead opted for system of weighted voting
Arguments for weighted voting

- Allows existing, natural community to be the unit of representation, regardless of size
- Avoids problem of periodic apportionments / gerrymandering
- Enables existing officeholders (mayor, etc.) to double as the community’s legislative representative
Argument against weights proportional to population

Banzhaf ("Weighted voting doesn’t work," *Rutgers Law Review* 1965) challenged the constitutionality of weighted voting in Nassau County, NY; at the time there were 115 total votes & 58 were required for passage

Nassau county breakdown:
Hempstead 1: 31
Hempstead 2: 31
North Hempstead: 28
Oyster Bay: 21
Glen Cove: 2
Long Beach: 2

Are any members of the board dummy voters?
*Iannucci v. Board of Supervisors* (1967)

NY’s highest court ruled that any weighted voting system in New York had to be based on Banzhaf’s method

- Implemented a computer-generated formula in Nassau and 23 other counties statewide that accounted for both population and the Banzhaf power of each board member

- Goal was to make population proportional to *voting power* as opposed to *voting weight*
A judicial decision based on findings of scholarly research

• A shift from (crude) focus on population equality to (sophisticated?) focus on legislator’s ability to determine public policy

“American courts have never clearly articulated exactly what it is they hope to equalize with [one man, one vote]... they have shown a zealous concern with something called ‘equality’ without coming to grips with the difficult notion of ‘representation.”

One problem with Banzhaf’s method

It does not take into account the empirical likelihood of different coalitions forming

- Unit domination: a small number of units can join together to make everyone else dummies

- Modified schemes: very large cities could have multiple representatives share voting weight; under Iannucci, only individual member power was calculated

- After Iannucci, Hempstead had 70 votes total, divided 35-35 between two members; each member had 27.8% of the weight, so Hempstead overall had 55.6% power and 56% of the population

- If one member had 70 votes, they would have had 88.9% of the power, and the system would have been unconstitutional
Measurement problems

How should the match-up between population share and power share be measured?

- Traditionally in SMDs: (1) find ideal district size; (2) find difference between largest and smallest districts and this ideal; (3) take each difference and divide by size of ideal to express discrepancy as % of ideal; (4) find the range of the discrepancy

- Example: ideal=50, smallest = 45, largest =60
  - Smallest yields difference of 5 under, largest is 10 over
  - 5/50 (10%) too small and 10/50 (20%) too large
  - Range of discrepancy is 30 percentage points
Application to weighted voting & Banzhaf indices

• (1) ideal power for each legislator is pop%; (2) find largest and smallest discrepancies from ideals; (3) take each difference and divide by size of ideal to express discrepancy as % of ideal; (4) find the range of the discrepancy

• Example: Smallest power discrepancy= 18% (ideal 20%), largest power discrepancy=11% (ideal 10%)
  – Smallest yields difference of 2% under, largest is 1% over
  – 2/20 (10%) too small and 1/10 (10%) too large
  – Range of discrepancy is 20 percentage points
In *Franklin v Krause (1973)* the court mistakenly changed its precedent for how it determines a discrepancy.

Smallest discrepancy: 18% actual (ideal 20%)
largest discrepancy: 11% actual (ideal 10%)

- Smallest yields difference of 2% under, largest is 1% over
- 2/20 (10%) too small and 1/10 (10%) too large
- Range of discrepancy is 20 percentage points
- Method of measurement in *Franklin* would have found discrepancy of 3 percentage points
In 1991 NYCLU filed suit that system violated one person, one vote

- 17 percent of residents were minorities and a third of registered voters were Democrats; almost every elected and appointed official in Nassau County was a white Republican
In a different 1989 case, NY Court ruled Banzhaf method failed because:

“It does not attempt to inquire whether, in terms of how a legislature actually works in practice, the districts have equal power to affect a legislative outcome. This would be a difficult and ever-changing task, and its challenge is hardly met by a mathematical calculation that itself stops short of examining the actual day-to-day operations of the legislative body.”

In 1991, Banzhaf argued Nassau misused his index. Since two members of six-person Board come from Hempstead and are Republicans who vote similarly on every issue, the application of the index confounds one of its main assumptions: that all winning coalitions are equally likely.
Voter power in large-scale elections

How should voter power be calculated in a presidential election, for example?

- In electoral college this is a two-step process
- First, every state’s power index is calculated; given its share of the 538 total electoral college votes, how often can it cast a pivotal collection of votes.
- Next, each voter within that state’s power index is calculated given the state population
- A voter’s total power over the election is calculated (roughly) as the fraction of times his vote could be pivotal in the presidential election
Large states better off

Voters in large states are helped by the electoral college (in terms of their power indices), because state votes are cast as a block

• Suppose that electoral voting weights perfectly correspond to “power”

• Suppose a group of 3 voters had a voting weight of 3 (as a group). Another group of 1 voter has voting weight of 1.

• There are 8 combinations of yes-no votes among the three voters

• Assuming majority rule in the group of 3, each voter is critical in 50% of those cases

• Each voter is critical for 50% of the block of three votes (1.5); The one voter with one vote is critical for 1 vote
### Appendix

#### Table 1

**Present Electoral College**

<table>
<thead>
<tr>
<th>State Name</th>
<th>Population Census 1960</th>
<th>Electoral Vote 1968</th>
<th>Relative Voting Power ($)</th>
<th>Percent Excess Voting Power (%)</th>
<th>Percent Deviation from Average Voting Power ($)</th>
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<td>Wisconsin</td>
<td>393177</td>
<td>12</td>
<td>1.788</td>
<td>78.8</td>
<td>6.2</td>
</tr>
<tr>
<td>Wyoming</td>
<td>330066</td>
<td>3</td>
<td>1.521</td>
<td>52.1</td>
<td>-9.6</td>
</tr>
</tbody>
</table>

(1) Includes the District of Columbia.

(2) Ratio of voting power of citizens of state compared with voters of the most deprived state.

(3) Percent by which voting power exceeds that of the most deprived voters (derivesions).

(4) Percent by which voting power deviates from the average of the figures to column 4.

Minus signs in Tables I-I indicate less than average voting power.

The “Banzhaf Fallacy,” Howard Margolis

Large states will necessarily look as though they have more “power” than small states when we take the likelihood of a vote to be a coin flip.

In reality, the most powerful states theoretically are generally not swing states (which are the most powerful in practice).

A large literature has looked at power indices empirically, to calculate a voter or state’s likelihood of being pivotal conditional on the empirical likelihood of various coalitions forming.
One last example

Consider weighted voting with weights \((5, 3, 1, 1, 1)\) and quota 8. The Banzhaf indices of the voters are

\[
\left(\frac{9}{19}, \frac{7}{19}, \frac{1}{19}, \frac{1}{19}, \frac{1}{19}\right)
\]

If Voter 1 gives one of his votes to Voter 2, to yield the new set of weights \((4, 4, 1, 1, 1)\) what are the new Banzhaf indices for the voters?