Today: More envy-free algorithms for n = 3
Cutting pie (n = 3)
Next Week: General envy-free algorithm, project feedback

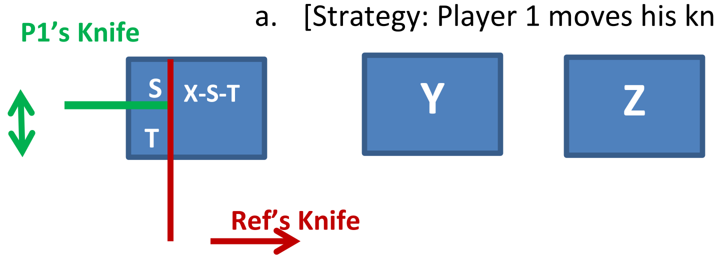
Recall the Stromquist Algorithm with 4 moving knives. The next algorithm reduces the number of knives.

**Levmore-Cook Algorithm**

2 moving knives: one automatic (held by the referee) and one held by a player

Rules:

1. Player 1 divides the cake into 3 pieces: x, y, and z
    a. [Strategy: Divide equally so that $v_1(x) = v_1(y) = v_1(z)$]
2. Players 2 and 3 declare one of the three pieces
    a. [Strategy: Declare the piece they view as largest]
3. If Players 2 and 3 declare different pieces, give each the piece declared and give Player 1 the remaining piece
    a. [Note: This makes the strategies accompanying Rules 1 and 2 compelling: Player 1 should divide the pieces equally as he may end up with any one of them. Players 2 and 3 should declare the pieces they view as largest as they may be assigned their declared pieces]
    b. In this case, we have proportionality and envy-freeness for each player
        i. It is as if Player 2 and Player 3 each pick first
4. If Players 2 and 3 declare the same piece (WLOG, let this piece be x) then the referee's knife moves across x from left to right while Player 1 is given the other knife. Player 1 moves the knife up and down, splitting the portion of x left of the referee's knife into two pieces, S and T. P1 can move his knife at will, while the referee's knife moves continuously.
    a. [Strategy: Player 1 moves his knife so that $v_1(S) = v_1(T)$]




5. At any point, Player 2 or Player 3 can say stop.
    a. [Strategy: PH says "stop" when $\max\{v_H(S \cup Y), v_H(T \cup Z)\} \geq v_H(X - S - T)$, justification provided below]
    b. Suppose PH said stop (Note PH is P2 or P3, never P1). Let PN be the other player. The pieces to be assigned are $S \cup Y, T \cup Z, and\ X - S - T$
    c. Pieces are chosen: PH chooses $S \cup Y\ or\ T \cup Z$, then P1 gets either $S \cup Y\ or\ T \cup Z$ (whichever was not chosen by PH) and PN gets X-S-T.

Strategy:

- P1 initially created x, y and z equal. While the ref's knife moves across x, x-s-t (weakly) diminishes in value to P1, so P1 needs to be sure that the eventual choice of $S \cup Y \text{ or } T \cup Z$ will be a tie.
  - P1 wants $v_1(S \cup Y) = v_1(T \cup Z)$, so moves his knife so that $v_1(S) = v_1(T)$
  - This makes the strategy for P1 outlined in step 4 compelling
- PH will choose from $S \cup Y \text{ and } T \cup Z$, so would only say "stop" if one of the following inequalities, which characterize the initial situation before the referee advances the knife across x and when S = T = 0, becomes false:
  - (a) $V_H(X - S - T) = v_H(X) > V_H(Y) = v_H(Y \cup S)$
  - (b) $V_N(X - S - T) = v_N(X) > V_N(Y) = v_N(Y \cup S)$
  - (c) $V_H(X - S - T) = v_H(X) > V_H(Z) = v_H(Z \cup T)$
  - (d) $V_N(X - S - T) = v_N(X) > V_N(Z) = v_N(Z \cup T)$
  - These four inequalities initially hold because P2 and P3 each prefer X to both Y and Z
  - PH says "stop" when $\max\{v_H(S \cup Y), v_H(T \cup Z)\} \geq v_H(X - S - T)$
    - This happens when inequality (a) or (c) becomes false
    - WLOG, suppose (a) is falsified first (or a tie occurs with (a) and (c) falsified simultaneously)
    - Then $V_H(X - S - T) \leq v_H(Y \cup S)$
    - As (c) still holds or was simultaneously falsified, $V_H(X - S - T) \geq v_H(T \cup Z)$
    - The previous two points imply that $V_H(Z \cup T) \leq v_H(Y \cup S)$
  - This provides the justification for PH's strategy in Rule 5.

Envy-Freeness

- PH picks $(Y \cup S)$ and achieves envy-freeness. The strategy given above shows that PH values this piece at least as highly as the other two.
- P1 gets the other piece, in this case $(T \cup Z)$. P1 maintained his knife so that $v_1(S \cup Y) = v_1(T \cup Z)$, so P1 never envies PH. $v_1(X - S - T) \leq v_1(T \cup Z)$, so P1 never envies PN.
- PN did not say "stop", so inequalities (b) and (d) still hold. Therefore
  - $V_N(X - S - T) > (Y \cup S)$
  - $V_N(X - S - T) > v_N(T \cup Z)$
  - So PN is envy-free

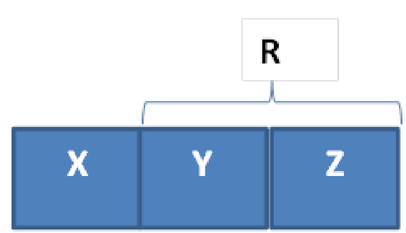
Possible Modification: Let PH choose either Y or Z and independently choose either S or T

This modification favors PN (since PH will call "stop" sooner in most cases)

**Webb Moving Knife Algorithm** (envy free for n = 3)

Rules:

1. P1, P2, and P3 perform Dubins-Spanier moving knife (WLOG suppose P1 says "stop" first and divides the cake into piece X and remainder R)

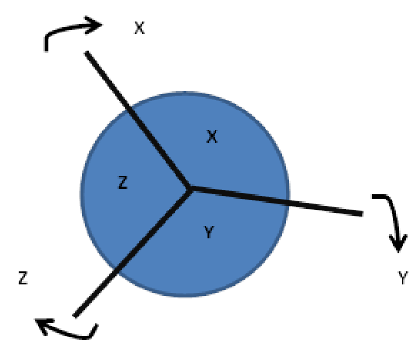


2. P1 and P2 perform Austin's Extension on the remainder (R), dividing it so that both agree the two portions, Y and Z, are of equal size
   a. $v_1(Y) = v_1(Z) = {}^{v_1(R)}/_2$
   b. $v_2(Y) = v_2(Z) = {}^{v_2(R)}/_2$
3. P3 chooses from X, Y, and Z (and is automatically envy-free as the first chooser)
4. P2 chooses. Since P2 didn't say "stop," $v_2(X) \leq {}^1/_3$ so $v_2(R) \geq {}^2/_3$ so $v_2(Y) = v_2(Z) \geq {}^1/_3$. P2 cannot envy since he is indifferent between Y and Z and (at least weakly) prefers each to X, and at least one of Y and Z must remain after P3 chooses
5. P1 chooses. By P1's strategy (given below), $v_1(R) = {}^2/_3$
   so $v_1(Y) = v_1(Z) = {}^{v_1(R)}/_2 = {}^1/_3$, making P1 envy-free

Strategy: P1 could get X, so should stay "stop" when $v_1(X) = \frac{1}{3} \rightarrow v_1(R) = \frac{2}{3} \rightarrow v_1(Y) = v_1(Z) = 1/3$

**Dividing Pie** (Brams, Taylor, and Zwicker) (envy-free for n = 3)

*How would you apply this same algorithm to cake?



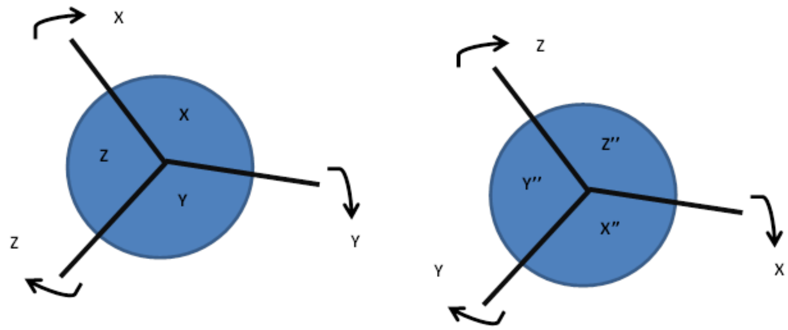

- 3 knives, rotating clockwise
- P1 controls the knives

Rules:
1. P1 moves the knives clockwise
2. P2 or P3 can say "stop" (WLOG suppose P2 says "stop" first)
3. Players choose in order P3, P2, P1

Strategy:
- P1: keep $v_1(X) = v_1(Y) = v_1(Z) = 1/3$, compelling since P1 could get any piece as the last to choose
- P2 and P3: say "stop" when two pieces are tied for largest (note: we need to show this is possible)

Envy-Free:
- P3 picks first, so is envy-free
- P2 felt there was a tie for the largest piece, so (at least) one of these remains and P2 is envy-free
- P1 kept the value of each piece equal, so is envy-free

Is a tie possible?




- As the knives rotate, when the X knife ends up in the position originally held up the Y knife, we must also have the Y knife in the position originally held by the Z knife and the Z knife in the position originally held by the X knife (since P1 keeps all pieces equal in his view)
- $Z'' = X, Y'' = Z, X'' = Y$
- WLOG, suppose P2 thought X had the uniquely largest value to begin with
  - Z started off inferior to X ($v_2(X) > v_2(Z)$)
  - After rotation, Z'' is the unique biggest piece ($v_2(Z'') > v_2(X'')$)
  - By the Intermediate Value Theorem, there must be some point when there existed pieces X' and Z' with $v_2(Z') = v_2(X')$
  - Just because a tie exists, does this imply that they are tied for largest?

* Show that either X' and Z' are tied for largest or at some point in time there is a tie between X' and Y' for largest