CSE 544  
10/11/12

Woodall:

\[ p, p_2 \text{ view at least 1 crumb differently} \]

\[
\begin{array}{c}
\text{crumb} \\
\bigtriangledown \\
\text{create 2 pieces}
\end{array}
\]

\[
\begin{array}{c}
p_1 \\
\bigtriangleup \\
p_2
\end{array}
\]

\[ p_1 \text{ thinks } A > B \]
\[ p_2 \text{ thinks } B > A \]

\[ \text{then strong proportional meaning both get } > \frac{1}{2} \]

Warning: \( X \) means something different from woodall here

\( X \) is a piece of cake (called \( S \) previously)

\[ V_1(X) < V_2(X) \]

so \( p, p_2 \) disagree about value of \( X \)

Goal in proving \( A + B \) exist:

1) Find \( p + q \) such that

\[ 0 \leq V_1(X) < \frac{p}{q} < V_2(X) \leq 1 \]

Cake

\[
\begin{array}{c}
|X| \\
\bigtriangleup \\
\text{rest of cake}
\end{array}
\]

disagree on \( C \)

2) \( p_2 \) cuts \( X \) into \( p \) equal pieces (of equal value to \( p_2 \))
3) $P_1$ cuts $r$ into $p$ pieces of equal value to $P_1$.

- $X_1, X_2, \ldots, X_p$ cut into $p$ pieces by $P_2$
- $r_1, r_2, \ldots, r_{Q-p}$ cut into $Q-p$ pieces by $P_1$

4 lemmas:

a) $\forall i \ V_2(X_i) > \frac{1}{Q}$

b) $\forall i \ V_1(R_i) > \frac{1}{Q}$

c) $\exists j \ V_2(R_j) < \frac{1}{Q}$

d) $\exists i \ V_1(X_i) < \frac{1}{Q}$

$P_1$ looks and sees all pieces $r_j > \frac{1}{Q}$

$P_2$ sees a piece with value less than $\frac{1}{Q}$ and sees all pieces $r_j > \frac{1}{Q}$

$P_1$ sees some piece as $\frac{1}{Q}$

Then $P_1$ sees $r_j > X_i$.

And $P_2$ sees $r_j < X_i$. The rest of Woodall's argument holds.

must prove lemmas
\[ 0 \leq V_1(x) \leq \frac{p}{q} \leq V_2(x) \leq 1 \]

a) \( \forall i \ V_2(x_i) > \frac{1}{q} \)

\[ V_2(x) \geq \frac{p}{q} \quad \text{given} \]

\[ \forall i \ p \cdot V_2(x_i) = V_2(x) \]

(\( p \) cuts \( x \) into \( p \) pieces of equal value)

\[ \forall i \ p \cdot V_2(x_i) > \frac{p}{q} \]

divide both sides by \( p \)

\[ \forall i \ V_2(x_i) > \frac{1}{q} \]

b) \( \forall i \ V_1(r_i) > \frac{1}{q} \)

\[ V_1(r) = 1 - V_1(x) \quad \text{(unit cake)} \]

\[ V_1(x) \leq \frac{p}{q} \quad \text{given} \]

\[ -V_1(x) > -\frac{p}{q} \quad \text{multiply by } -1 \]

\[ 1 - V_1(x) > 1 - \frac{p}{q} \quad \text{add } 1 \]

\[ V_1(r) > 1 - \frac{p}{q} \quad \text{substitute from above} \]

\[ V_1(r) > \frac{a - p}{q} \]

\[ \forall i \ (a - p) \cdot V_1(r_i) = V_1(r) \cdot p, \text{divided } r \text{ into } a - p \text{ pieces} \]

\[ \forall i \ (a - p) \cdot V_1(r_i) > \frac{a - p}{q} \]

\[ \forall i \ V_1(r_i) > \frac{1}{q} \]
c) \( \exists j \) \( V_2(r_j) < \frac{1}{Q} \)

suppose not (each piece \( \geq \frac{1}{Q} \))

then \( V_i \geq V_2(r_j) \geq \frac{1}{Q} \)

with \( Q - p \) pieces of \( r \)

\( V_2(r) \geq (Q - p) \frac{1}{Q} \)

\( V_2(r) \geq \frac{Q - p}{Q} \)

\( V_2(r) \geq 1 - \frac{p}{Q} \)

from a \( v_i \) \( V_2(x_i) > \frac{1}{Q} \)

with \( p \) pieces of \( x \)

\( V_2(x) > \frac{p}{Q} \)

value of whole cake is

\( V_2(x) + V_0(r) > \left( \frac{p}{Q} \right) + \left[ 1 - \frac{p}{Q} \right] \)

which is \( > 1 \) so we get a contradiction

How hard is it to find \( P + Q \)

\( V_1(x) = \frac{1}{101} \)

\( V_2(x) = \frac{1}{100} \)

\( 0 \leq V_1(x) < \frac{p}{Q} < V_2(x) \leq 1 \)

\( \frac{1}{101} < \frac{p}{Q} < \frac{1}{100} \)

suppose \( V_1(x) = \frac{a_1}{b_1} \) \( V_2(x) = \frac{a_2}{b_2} \)

how big are smallest \( P + Q \)?
questions

- what if \( V_1(x), V_2(x) \) are irrational
- # of cuts isn't known a priori
- still bounded
- adopt this to dividing a list of items
- is there value to having \( x \) as large as possible

Envy free for 3 people

Let \( \rho_1 \) makes \( X_1, X_2, X_3 \)
\( \rho_2 \) declares acceptable set
\( \rho_3 \) does same

not envy free

Conway Guy Selfridge

\( \rho_1 \) creates 3 equal pieces \( X_1, X_2, X_3 \)
\[ V_1(X_1) = V_1(X_2) = V_1(X_3) = \frac{1}{3} \]
in the end, if \( \rho_1 \) gets one of these pieces and the other 2 don't change then \( \rho_1 \) has no envy

2) \( \rho_2 \) sorts them
\[ V_2(X_1) \geq V_2(X_2) \geq V_2(X_3) \]

3) \( \rho_2 \) trims \( X_1 \) if necessary so \( \rho_2 \) \( V_2(X_1') = V_2(X_2) \)
now \( V_2(X_1') = V_2(X_2) \geq V_2(X_3) \)
4) P3 chooses first from X1', X2, X3 and therefore cannot envy.
   P2 must pick X2 if available, otherwise take any.
   P1 chooses from X2 + X3 gets what's left from X2 or X3. X1' must have been taken already.

What about trimming (E)?
- Use this algorithm recursively, stop when E is really small.

Or turns out we know more about E than we knew about the whole cake.
P2 or P3 received X1', call the person who got X1' P1.
P1 cannot envy P1 even if P1 got all of E. (P1 has an irrevocable advantage over P1)

Players
- P1 - initial divider
- P1 - got X1'
- Pn - other person

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unbounded
1) $P_n$ divides $E$ into 3 equal pieces
   $E_1, E_2, E_3$

2) $P_T$ chooses one therefore envy free
   because $P_T$ chooses first

3) $P_1$ chooses second (P, cannot envy $P_T$ because)
   irreducible advantage

4) $P_n$ gets last piece (no envy because $P_n$)
divided

Stromquist moving knives

1) ref moves one knife across cake
2) 3 players each get a knife it can be
   anywhere over the cake, to right of ref
3) when a player says stop (player i)
   $P_i$ gets the piece to left of refs knife.
   Simultaneously, middle players knife cuts
   Player other than i whose knife is
   closest to refs knife gets $Y$, other
   gets $Z$