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CSE 544 T  prop - proportional

Exam I, Oct 8 (Monday)
open notes, closed internet. Look at * questions

So far:
* Divide & Chase, n=2
* Dubins-Spanier many knife*, prop n people
  * Austin moving knives, 2 parties decide "1/2"
    - 2 parties decide m "1/m"
    - 2 parties divide cake into m "equal" pieces

* Branch-Knaster trimming *
  Fink's line chooser
    - n parties, dynamic
  Woodall's strongly prop, n=2
  Steinhaus-Kuhn lone divider prop n=3
  Conway-Guy-Selﬁdge, n=3 envy-free

Fink line chooser:

\[ \begin{array}{c}
\text{cake divided} \\
\text{new person joins,}
\text{redistribute the cake}
\end{array} \]

1) divide cake into 2.
\[ \begin{array}{c}
\text{O new person}
\text{selects a parent piece from each.}
\end{array} \]
Finik's lone chaser

Start $n = 2$ [divide & choose]

$P_1$ creates $X_1, X_2$ where $v(X_1) = v(X_2) = \frac{1}{2}$

Suppose $P_2$ chooses, take $\pi_2$

$P_1$ gets $\pi_1$

Standard divide & choose

Along comes $P_3$

$\begin{align*}
P_{11} & \quad \pi_{12} & \quad \pi_{13} \\
\pi_{21} & \quad \pi_{22} & \quad \pi_{23}
\end{align*}$

Equal pieces to $P_1$

Equal pieces to $P_2$

$P_1 \quad v(\pi_{11}) = v(\pi_{12}) = v(\pi_{13}) = \frac{1}{2} \left( \frac{1}{2} \right) = \frac{1}{4} \quad \text{to} \quad P_1$

$P_2 \quad v(\pi_{21}) = v(\pi_{22}) = v(\pi_{23})$, each $\geq \frac{1}{4} \quad \text{to} \quad P_2$

$P_3$ gets to pick one piece from $\pi_{11}^*$

and one piece from $\pi_{22}^*$

One piece from $P_1$, and one from $P_2$

$P_3$ takes one piece from $P_1$ so

the

value of the remaining cake $= \frac{2}{6} = \frac{1}{3}$, $n = 3$

so this is proportional.
For \( n = 2 \) the same value left to \( \pi_1 \) is \( \frac{1}{2} \). For \( n = 3 \), proportional.

What about \( \pi_3 \)'s value of the cake?

\[
\pi_1 \text{ 's piece} \quad \pi_2 \text{ 's piece} \\
\text{value } \pi_3 = \alpha \quad \text{value of } \pi_3 = 1 - \alpha
\]

If the whole piece of cake has value \( \alpha \)

Then \( \frac{1}{3} \) to the divider.

Not all 3 pieces can have value \( \leq \frac{1}{3} \).

\( \pi_3 \) gets at least \( \frac{1}{3} \alpha \) and \( \frac{1}{2} \alpha = 1 \)

\[
\frac{1}{3} \alpha + \frac{1}{3} (1 - \alpha) = \frac{\alpha + 1 - \alpha}{3} = \frac{1}{3}
\]

\( \Rightarrow \pi_3 \) picks up cake from \( \pi_1 + \pi_2 \)

with a value of at least \( \frac{1}{3} \) from \( \pi_3 \)’s perspective.

Along comes \( \pi_4 \) divide cake into 4 pieces.

\[
\pi_1 \quad \pi_2 \quad \pi_3
\]

This is not envy-free.
The number of cuts made is $O(n^3)$ cuts.

$n = \begin{align*}
2 & \quad 1 \\
3 & \quad 5 = 4 + 1 \\
4 & \quad 14 = 9 + 4 + 1 \\
\end{align*}

Woodall's algorithm

- Strongly planar
- related literature
  - Problem of the Nile, 1938 - 1961
  - Ham Sandwich theorem, 1942 - 1985

The cakes value must differ to the two parties.

Assume cake $K$ divided into $\Box$. People $v_1(s) \neq v_2(s)$.

Dif value 2 people $v_1(s) > v_2(s)$.

* From the above, show that it is possible from $K$ to create $\Box \Box$ when $x_1$ and $x_2$ are not necessarily the whole cake.

Such that $x_1$ values $V_1(x_1) > V_1(x_2)$

$x_2$ values $V_2(x_1) < V_2(x_2)$
Problem

Given \( x_1, x_2, r \) for rest of the cake,

Assign cake \( \Pi_1, \Pi_2 \) such that:

\[
V_1(\Pi_1) > \frac{1}{2} \\
V_2(\Pi_2) > \frac{1}{2}
\]

Already have \( p_1 \) likes \( x_1 \) and \( p_2 \) likes \( x_2 \) and

the rest at the centre, \( r \).

\( p_1 \) and \( p_2 \) simply play divide and chase
in \( r \), creating \( v_1 + v_2 \).

Analysis for person 1 (\( p_1 \))

Suppose \( V_1(x_1) = \alpha_1 \), \( V_1(x_2) = \alpha_2 \)

\[ V_1(r) = 1 - \alpha_1 - \alpha_2 \]

\[
\begin{array}{ccc}
|x_1| & x_2 & r \\
p_1 & | & p_2 \\
\end{array}
\]

By divide and chase \( p_1 \) gets \( r_1 \), \( p_2 \)
gets \( r_2 \) and \( v(r_1) > \frac{1 - \alpha_1 - \alpha_2}{2} \) for \( p_1 \)

Then \( p_1 \) takes \( x_1 \) and \( p_2 \) takes \( x_2 \).

\( p_1 \)’s value

\[
\begin{align*}
V(x_1) &= \alpha_1 \\
V(r_1) &\geq 1 - \frac{\alpha_1 + \alpha_2}{2} \\
V(x_1 + r_1) &\geq \alpha_1 + \frac{1 - \alpha_1 - \alpha_2}{2} = \frac{2\alpha_1 + 1 - \alpha_1 - \alpha_2}{2} \\
V(x_1 + r_1) &\geq \frac{1}{2} + \frac{\alpha_1 - \alpha_2}{2} \\
\end{align*}
\]

\( \therefore V > \frac{1}{2} \) because \( \alpha_1 > \alpha_2 \)
Skiplines - Kuhn

Condiver, complicated

0. $P_1$ creates $x_1, x_2, x_3$ where $v(x_1) = v(x_2) = v(x_3) = \frac{1}{3}$ from

1. $P_2$ defines $S_2 = \{x_1, x_2, x_3\} \mid \forall x \in S_2 \quad v_2(x) \geq \frac{1}{3}$

[Lemma $|S_2| \geq 1$]

2. $P_3$ does the same thing to create $S_3$ using $x_1, x_2, x_3$

Two possibilities

0. a) Lucky $|S_2| > 1$ , Assign from smallest to largest

$|S_2|, |S_3|$ size.

Ex. $S_2 = \{x_1, x_3\} \quad S_3 = \{x_3\}$

$P_3 = x_3 \quad P_2 = x_1$ and pass

1 gets whatever is left over because they like every piece equally.

0. b) Unlucky $|S_2| = |S_3| = 1$

There must be an unacceptable piece (say $x_1$) such that $v_2(x_1) < \frac{1}{3} \quad v_3(x_1) < \frac{1}{3}$, so
give $x_1$ to $P_1$. Then combine $x_2 + x_3$

and play divide and choose with $P_2 + P_3$.

Not envy free. $P_1$ can envy in case b. divide and choose

In case a $P_2$ can envy $P_3$ but not

$P_1$. $P_1$ cannot envy anyone in a because he values each piece equally.