So far,
we can reduce the items to
those in dispute, use balanced
alternatives to allocate the
disputed items

But envy can result

Balanced alternative is
envy-free item-by-item in the limit

But not equitable - we don't even
know how an item is valued by either
player

Envy-free, efficient, equitable all at once?

Yes \( n = 2 \) adjusted winner

Point allocation schemes

Example 1) Leng-Epstein 2) Saltz
arms reduction

1) We allocate 1,000 pts across
adversary's weapons, they destroy 10%

2) We allocate 1,000 pts across our
weapons, they decide which to
destroy to get 10% reduction
These are like Divide & Choose
One side distributes points not knowing how the other side will choose

AW compels truthful valuation, as we will see

Example: Ann & Ben divorce (again).

<table>
<thead>
<tr>
<th>Item</th>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>Retirement Account</td>
<td>50</td>
<td>40</td>
</tr>
<tr>
<td>Home</td>
<td>20</td>
<td>30</td>
</tr>
<tr>
<td>Summer cottage</td>
<td>15</td>
<td>10</td>
</tr>
<tr>
<td>Investments</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>Misc</td>
<td>5</td>
<td>10</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td>100</td>
<td>100</td>
</tr>
</tbody>
</table>

Step 1: Award items to higher bidder

Step 2: Compute scores
A: 50 + 15 = 65
B: 30 + 10 = 40

Step 3: Use ties to increase equitability
B gets Investments = 10
B has $50
That was an adjustment — transfer from one to the other to move toward equitable allocation.

In general, we look to transfer $x$ from initial winner to initial loser.

We measure

$$r(x) = \frac{V_w(x)}{V_l(x)}$$

(In a tie, $r_x = 1$. Generally $r_x > 1$)

$$r(\text{Retirement Acct}) = \frac{50}{40} = 1.25$$

$$r(\text{Summer Caching}) = \frac{15}{10} = 1.5$$

We transfer using smallest $r(x)$ first.

So, we start with the retirement account.

But if we give it all to Ben:

A: $65 - 50 = 15$ too much!

B: $50 + 40 = 90$
Ideally, the account can be split to make Ann and Ben's items equitable.

Let \( x \) = fraction of account transferred

\[
65 - 50x = 50 + 40x
\]

\[
x = \frac{1}{6}
\]

\( \frac{1}{6} \) of Retirement Account goes from Ann to Ben.

A: \( 60 - \frac{1}{6} \times 56 = 56.69 \)
B: \( 50 + \frac{1}{6} \times 46 = 56.67 \)

Equitable!

In general, the following:

1) Each party is given the same number of points.

   * What happens if different points are allotted

2) Parties independently allocate points to items.

3) Initially assign items as if by auction, using points.

4) Ties (really just special case of 5) Award to party with fewer points, one-by-one.
5) Reassign items from higher to lower party, from smallest to largest ratio of valuation why smallest to largest?

What do we do with indivisible items?

<table>
<thead>
<tr>
<th>Item</th>
<th>p_1</th>
<th>p_2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Boat</td>
<td>6</td>
<td>14</td>
</tr>
<tr>
<td>Motor</td>
<td>6</td>
<td>14</td>
</tr>
<tr>
<td>Piano</td>
<td>17</td>
<td>2</td>
</tr>
<tr>
<td>Computer</td>
<td>17</td>
<td>1</td>
</tr>
<tr>
<td>Rifle</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>Tools</td>
<td>6</td>
<td>2</td>
</tr>
<tr>
<td>Tractor</td>
<td>2</td>
<td>21</td>
</tr>
<tr>
<td>Track</td>
<td>8</td>
<td>14</td>
</tr>
<tr>
<td>Moped</td>
<td>17</td>
<td>14</td>
</tr>
<tr>
<td>Moped</td>
<td>17</td>
<td>14</td>
</tr>
</tbody>
</table>

100  100

\[ \text{74 - 17x = 67 + 14x} \]

\[ x \approx 0.226 \]
\[ P_2 \text{ gets } 23\% \text{ of a moped} \]
\[ \text{Must sell and use cash. Whichever}\]
\[ \text{value, } 77\% \text{ goes to } P_1; \]
\[ 23\% \text{ goes to } P_2. \]

* How would balanced allocation have allocated these, and what is the result? (Rank order: reveal, award, defer, alternate)

* AW is
  1) Envy-free (need only show prop, \( n = 2 \))
  2) Efficient!
  3) Equitable

Manipulable?

<table>
<thead>
<tr>
<th></th>
<th>A</th>
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</tr>
</thead>
<tbody>
<tr>
<td>X</td>
<td>75</td>
<td>25</td>
</tr>
<tr>
<td>Y</td>
<td>25</td>
<td>75</td>
</tr>
</tbody>
</table>

AW gives each 75 nice!

Suppose A knows B's points, but B doesn't know A's

<table>
<thead>
<tr>
<th></th>
<th>A'</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>X</td>
<td>26</td>
<td>25</td>
</tr>
<tr>
<td>Y</td>
<td>74</td>
<td>75</td>
</tr>
</tbody>
</table>
A gets X, as before  
B gets Y  

A: 26  
B: 75  

Some value of Y must transfer from B to A  

\[ 26 + 74x = 75 - 75x \]  
\[ x \approx .33 \]  

A: 26 + \frac{1}{3} 74 \approx 50  
B: 75 - \frac{1}{3} 75 \approx 50  

But A really gets  

\[ 75 + \frac{1}{3} 25 \approx 83.33 \]  

If both knew the others’ parts, they’d try this  

<table>
<thead>
<tr>
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<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>X</td>
<td>26</td>
<td>\frac{74}{1}</td>
</tr>
<tr>
<td>Y</td>
<td>\frac{74}{1}</td>
<td>26</td>
</tr>
</tbody>
</table>

A gets Y, really 25 pts? Inefficient  
B gets X, really 25 pts? Unstable
So insincerity can bring risk

Extension beyond \( n = 3 \)

Theorem cannot satisfy necessarily all 3 properties (efficient envy-free equitable)

[Reijnierse & Potters]

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x )</td>
<td>40</td>
<td>30</td>
<td>30</td>
</tr>
<tr>
<td>( y )</td>
<td>50</td>
<td>40</td>
<td>30</td>
</tr>
<tr>
<td>( z )</td>
<td>10</td>
<td>30</td>
<td>40</td>
</tr>
</tbody>
</table>

\( A - X \) \( B - Y \) \( C - Z \)

Efficient + equitable

Not envy free

\[ v_X(A) = 50 > v_Y(B) \]

If they swap

\( A - Y \) \( B - X \) \( C - Z \)

Not equitable + B envies A