Properties of TMs' languages

\[ P = \{ e(t) \mid p(L(t)) \} \]

Examples

\[ P_1 = \{ e(t) \mid |L(t)| \leq 5 \} \]

TMs whose languages contain no more than 5 strings.

\[ P_2 = \{ e(t) \mid e(\text{dog}) \in L(t) \} \]

TMs whose languages contain the (encoded) string "dog".

A property is itself a language whose strings are TMs that possess the desired property. Are her properties recursive?

\[ e(t) \rightarrow \text{TM for } P \]

Does T have property \( P \)?

Y N
Rice's theorem:
Any non-trivial property of
the RE languages (TMs)
is non-recursive.

A property $P$ is TRIVIAL if

$P = \emptyset$: No TM can satisfy $P$.

$P = \{ \text{all TMs} \}$: Every TM satisfies $P$.

$P = \{ e(t) \mid L(t) \leq \varepsilon^* \}$: All TMs.

$P = \{ e(t) \mid L(t) > \varepsilon^* \}$: No TM.

The non-trivial part is needed for the proof...

Proof: We show $P$ recursive $\Rightarrow$ SA recursive.

Consider any non-trivial property $P$.

If $P$ is recursive, we have $e_P$ as
a TM that decides membership in $P$.

\[
\begin{array}{c}
e(t) \\
\downarrow \\
v
\end{array} \xrightarrow{T_P} \begin{array}{c}
\text{as 258.1}
\end{array}
\]
Because $P$ is nontrivial, we know that among all TMs we can find

$$T, T' \in P$$

$$T \neq T' \in P.$$  

Also let's define

$$\text{Empty} = \{ e(T) \mid L(T) = \emptyset \}$$

These TMs accept no string. But one or more may satisfy one property $P$.

Recall $P_1 = \{ e(T) \mid |L(T)| \leq 5 \}$

Surely $\text{Empty} \leq P_1$

Surely $\text{Empty} \cap P_2 = \emptyset$ if a TM's language is $\emptyset$ it can't contain "dos"
Proof

Case 1: Empty \( \cap \) P = \( \emptyset \)

\[ L(T_{\text{obs}}) = \emptyset \quad \text{if} \quad T \text{ does not accept } w \]

In this case, \( T_{\text{obs}} \subseteq P \)

So \( \Rightarrow \) say No

\[ L(T_{\text{obs}}) \text{ satisfies } P \quad \text{if} \quad T \text{ accepts } w \text{ in } T_{\text{obs}} \]

So \( \Rightarrow \) say Yes
Case 2: Empty $\leq P$

Case 1 proof fails because I would say yes to all $T_0S$, even if $T$ does not accept $w$.

So...

Does $T$ not accept $w$?

$L(T_0S) = \emptyset$ if $T$ does not accept $w$.

In this case, $T_0S \in P$.

So $\Rightarrow$ accepting $YES$.
$L(TOBS) \text{ does not satisfy } P$

If $T$ accepts $w$

$TOBS$ behave like a TM not in $P$

so $\not\exists \text{ supp } NO$