HW 6

Due: October 23rd, 2025

Problem 1:

To analyze the number of steps until our coloring Markov chain converges, we defined a distance D_t between the pair of states of our coupling after t steps, and then showed that when $D_t > 0$ there exists a $\beta < 1$ such that

$$E[D_{t+1}|D_t] \leq \beta D_t.$$

Part A.

Under this condition, give a general upper bound for the number of steps until the distribution on states is ϵ-close to the stationary distribution in terms of β and D^*, where D^* is the maximum distance over all possible pairs of initial states for the coupling.

Part B.

Suppose that instead we have

$$E[D_{t+1}|D_t] \leq D_t$$

(i.e., taking $\beta = 1$). Suppose we have the additional conditions that D_{t+1} is one of D_t, $D_t - 1$, or $D_t + 1$ and that $\Pr(D_t \neq D_{t+1}) \geq \gamma$. Give an upper bound for the number of steps until the distribution on states is ϵ-close to the stationary distribution in terms of ϵ, D^*, and γ. Your answer should by polynomial in D^* and $1/\gamma$. (*Hint:* Think of D_t as being similar to a random walk on a line graph.)

Part C.

Using (b), show that the mixing time of the coloring Markov chain given in lecture is polynomial in the number of vertices in the graph and $\ln(1/\epsilon)$, even when the number of colors is only $4d$ (rather than at least $4d + 1$) when the graph has maximum degree d.

Problem 2: Max-Flow, Min-Cut, and LP Duality

Consider a *directed* graph $G = (V, E)$, with a *source* vertex $s \in V$ and a *sink* vertex $t \in V$. Each edge $(u, v) \in E$ has a non-negative integer *capacity* $c(u, v) \geq 0$. A **flow** is a function $f : E \to \mathbb{R}$ satisfying:

1. **Capacity Constraint:** For all $(u, v) \in E$, $0 \leq f(u, v) \leq c(u, v)$.
2. **Flow Conservation:** For all $u \in V \setminus \{s, t\}$,

$$\sum_{(v,u)\in E} f(v, u) = \sum_{(u,w)\in E} f(u, w)$$

   (i.e., the total flow into a vertex equals the total flow out of it).

3. **Source and Sink Constraints:** no flow re-enters the source, i.e.,

$$\sum_{(v,s)\in E} f(v, s) = 0$$

   and no flow exits the sink, i.e.,

$$\sum_{(t,v)\in E} f(t, v) = 0.$$

Part A. LP Formulation for Maximum Flow

The total value of a flow, denoted $|f|$, is the total flow leaving the source s:

$$|f| = \sum_{(s,u)\in E} f(s, u)$$

Your task is to formulate a linear program to find the maximum possible flow value. The variables in your linear program should be the flow values f_{uv} for each edge $(u, v) \in E$. The formulation must include the capacity constraints and flow conservation constraints for all relevant vertices.

Part B. The Dual LP and the Minimum Cut

Recall that an **s-t cut** is a partition of the vertices V into two sets, S and $\bar{S} = V \setminus S$, such that $s \in S$ and $t \in \bar{S}$. The **capacity** of the cut $(S, \bar{S})$ is the sum of the capacities of all edges that start in S and end in $\bar{S}$: $C(S, \bar{S}) = \sum_{u\in S, v\in\bar{S}} c(u, v)$. The min-cut problem seeks to find an s-t cut with the minimum possible capacity.

1. Write down the dual of the max-flow linear program you formulated in part (a).

2. Show how this dual linear program relates to the min-cut problem. Specifically, show that if the dual variables corresponding to the vertices are restricted to be integers, the dual linear program is equivalent to the problem of finding a minimum s-t cut in the graph.