HW 4

Due:September 25th, 2025

Problem 1: Best Response and Linear Programming

We are given a two-player zero-sum game with a payoff matrix $A \in \mathbb{R}^{m \times n}$, where the row player chooses a mixed strategy $\vec{x} \in \mathbb{R}^m$, and the column player chooses a mixed strategy $\vec{y} \in \mathbb{R}^n$. The row players payoff is given by $\vec{x}^\top A\vec{y}$. Since this is a zero-sum game the column player is attempting to maximize this value, and the row player is attempting to minimize it.

Part A.

Let the row player fix a mixed strategy $\vec{x} \in \Delta_m$ (a distribution over all of their possible options). Give a linear program such that the set of optimal solutions of the linear program is precisely the set of best responses for the column player. Explain why these optimal solutions are precisely the column player's possible best responses to a fixed $\vec{x}$.

Part B.

Conversely, now, consider a linear program $A\vec{x} \geq \vec{b}$ s.t. $\vec{x} \geq \vec{0}$ and $\|\vec{x}\|_1 = 1$ where $\vec{b} \geq \vec{0}$. Describe a payoff matrix for a zero-sum game such that the value of the game (for the row player) is nonnegative iff the linear program has a feasible solution. Specifically prove that when the linear program is feasible, the value is nonnegative and that you can compute a feasible solution to the linear program in linear time from optimal mixed strategies of the game.

Part C.

Continuing the above, prove that if the linear program of the above form is not feasible, then there exists a vector $\vec{w} \geq \vec{0}$ such that for all $\vec{x} \geq 0$ with $\|\vec{x}\|_1 = 1$, $\vec{w}^\top A\vec{x} < \vec{w}^\top \vec{b}$. (Recall that when we solved linear programs with the multiplicative weights algorithm, we were using the weights to find such a $\vec{w}$.)

Problem 2: Graph Vertex Expansion

In lecture, we defined *conductance* of a graph, which is a measure of how well-connected the graph is. Cheeger's inequality shows that the conductance is closely related to the difference between the largest and second-largest eigenvalue. Another closely related quantity is the "vertex expansion":[1] For an undirected graph $G = (V, E)$ and any set $S \subseteq V$, let $N(S) = \{u \in V : (u, v) \in E \text{ for some } v \in S\}$ be the *neighbors* of S. A graph has (K, A)-*vertex expansion* if $\forall S \subset V, |S| \leq K, |N(S)| \geq A|S|$. In this problem, you'll related vertex expansion to the second-largest eigenvalue.

For the argument, we'll consider the "collision probability" of a distribution on vertices i.e., the probability that when we independently sample twice from the distribution, we get the same vertex. For a probability distribution π over $[n]$, the collision probability is given by:

$$||\vec{\pi}||_2^2 = \sum_{i=1}^{n} \pi_i^2 \tag{1}$$

Part A.

Prove that $||\vec{\pi}||_2^2 \geq \frac{1}{|S(\vec{\pi})|}$ where $S(\vec{\pi}) = \{i : \pi_i > 0\}$

Part B.

Prove that $||\vec{\pi}||_2^2 = ||\vec{\pi} - \frac{\vec{1}}{n}||_2^2 + \frac{1}{n}$ where $\frac{\vec{1}}{n}$ is the uniform distribution

Part C.

Suppose that every vertex in our undirected graph G has degree d, so that the transition matrix of a random walk on G is given by the adjacency matrix scaled by $1/d$. If there is a constant $\lambda < 1$ such that the transition matrix of G has second-largest eigenvalue $\lambda_2 \leq \lambda$ then for any $\alpha < 1$, G has $\left(\alpha n, \frac{1}{(1-\alpha)\lambda^2+\alpha}\right)$-vertex expansion.

[1] Conductance is sometimes called "edge expansion".