HW 3

Due: September 18th, 2025

Problem 1: Sampling for Path Optimization

You are given a directed graph $G = (V, E)$ representing a road network. Suppose that over T days, you repeatedly need to travel from a point $u \in V$ to $v \in V$. On each day, the edges may take a varying amount of time to traverse. We represent the time to travel the edges by positive edge weights $w_e^{(t)} > 0$ for $t = 1, \ldots, T$. For a simple (loop-free) path from u to v, $P_{u,v}$, the weight of this path on day t is naturally defined as:

$$W^{(t)}(P_{u,v}) = \sum_{e \in P_{u,v}} w_e^{(t)} \tag{1}$$

Of course, you would like to minimize the amount of time you spend traveling.

Suppose you have access to a polynomial-time sampling algorithm, $\mathcal{O}$, which, when given some set of weights W for G and two vertices $u, v \in V$, returns a simple (loop-free) path $P_{u,v}$ with probability proportional to $\exp(-\eta W(P_{u,v}))$ for a fixed parameter $\eta > 0$. This means that each path has a probability given by:

$$Pr[P_{u,v}] = \frac{\exp(-\eta W(P_{u,v}))}{Z}, \text{where } Z = \sum_{\text{simple path } P'_{u,v}} \exp(-\eta W(P'_{u,v})) \tag{2}$$

Part A.

Describe a polynomial-time algorithm (using calls to $\mathcal{O}$) based on the Multiplicative Weights method to choose paths for each day. On each day, your algorithm will output a single path sampled from a distribution over paths, such that the expected weight will be nearly as close as possible to the best single fixed path $P^*_{u,v}$ (to be used on all days). Your algorithm should run in time polynomial in T and the size of the graph.

Part B.

Prove that your algorithm gives an output with expected total travel time $E[\sum_{t=1}^{T} W]$ such that:

$$E[\sum_{t=1}^{T} W] \leq (1+\epsilon) \sum_{t=1}^{T} W^{(t)}(P^*_{u,v}) \tag{3}$$

Where $P^*_{u,v}$ is the optimal (minimum weight) path from u to v. Give a bound on the number of rounds needed (T) that is polynomial in terms of ϵ, the number of edges $|E|$, and the maximum weight M of any simple path from u to v.

Problem 2: Random Walk over Edges

This problem is a variation of HW1 problem 2. Given an undirected graph $G = (V, E)$ we will define a random walk over the edges instead of the vertices. So, at time step $t = 1, 2, \ldots$ the walk will choose an adjacent edge to travel to. The state space will be defined as the set of edges:

$$S = \{(u, v) \in V \times V : (u, v) \in E\} \tag{4}$$

This represents traveling from vertex u to vertex v along edge (u, v). At the next step walker will uniformly sample a next edge $(v, w) \in E$. This gives a transition matrix:

$$P\big((u,v),(v,w)\big) = \begin{cases} \frac{1}{\deg(v)} & \text{if } (v,w) \in E \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

Describe a vector v such that $Pv = v$ (and verify this relation holds). Show that this is the "stationary distribution" of the random walk.