CSE 559A: Computer Vision

Fall 2018: T-R: 11:30-1pm @ Lopata 101

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http://www.cse.wustl.edu/~ayan/courses/cse559a/

Sep 18, 2018

ADMINISTRIVIA

- Recitation will be this Friday (9/21) in Jolley 309.
  - Will go over topics relevant to Pset.
- Problem Set due next Tuesday

GENERAL

Notes on Parallelization

Say your simple code looked like this

```
for i in Indices:
    y = func1(x[i])
    z = func2(x[ifunc(i)])
    p = func3(y,z)
    out[i] = func4(p)
```

Replace as

```
y = func1(x[Indices])
z = func2(y[ifunc(Indices)])
p = func3(y,z)
out[Indices] = func4(p)
```

- Most numpy functions that act on single numbers can be used elementwise on arrays.
- Think about all the steps you would do for each number in a loop: Can these steps can be carried out independently for different loop indices ?
- If so, replace them with array operations.

IMAGE RESTORATION

$$\hat{X} = \arg\min_{X} \|X - Y\|^2 + (X - \mu)^T \Sigma^{-1} (X - \mu)$$

where, $\mu = W^T \mu_c, \Sigma = W^T D_c W.$

The solution is:

$$\hat{X} = W^T (I + D_c^{-1})^{-1} W \quad (Y + W^T D_c^{-1} \mu_c)$$

How would you code this up ?

IMAGE RESTORATION

$$\hat{X} = W^T (I + D_c^{-1})^{-1} W \quad (Y + W^T D_c^{-1} \mu_c)$$
$$= {\color{red} W^T} \quad (I + D_c^{-1})^{-1} (WY + D_c^{-1} \mu_c)$$

```
xc = wvlt(Y)
xc += mu_c / sigma2c
xc /= (1 + 1/sigma2c)
X = invwvlt(xc)
```

Note that here `mu_c` and `sigma2c` are arrays the same size as `xc`.

IMAGE RESTORATION

More general optimization setting:

$$\hat{X} = \arg\min_X D(X; Y) + R(X)$$

$$\hat{X} = \arg\min_X \sum_n \|X[n] - Y[n]\|^2 + \lambda \sum_n \left(((G_x * X)[n])^2 + ((G_y * X)[n])^2\right)$$

Let A_x and A_y be matrices corresponding to convolution with G_x and G_y.

$$\hat{X} = \arg\min_X \|X - Y\|^2 + \lambda \ \left(\|A_x X\|^2 + \|A_y X\|^2\right)$$

$$= \arg\min_X (X - Y)^T (X - Y) + \lambda X^T (A_x^T A_x + A_y^T A_y) X$$

Let's change this to our standard quadratic form:

$$= \arg\min_X \quad X^T \ \left(I + \lambda(A_x^T A_x + A_y^T A_y)\right) \ X - 2X^T Y + Y^T Y$$

$$\text{And so,} \quad \hat{X} = \left(I + \lambda(A_x^T A_x + A_y^T A_y)\right)^{-1} \ Y$$

How do you do this matrix inverse ?

IMAGE RESTORATION

$$\hat{X} = \left(I + \lambda(A_x^T A_x + A_y^T A_y)\right)^{-1} \; Y$$

Remember, (circular) convolution is diagonalized in the Fourier domain !

$$A_x = S\, D_x\, S^*$$

- Multiplication by S is the inverse Fourier transform
- Multiplication by S^* is the forward Fourier transform
- $SS^* = I$
- D_x is a diagonal matrix with the Fourier transform coefficients of G_x

$$A_x^T A_x = A_x^* A_x = S\, D_x^* D_x S^* = S\, \|D_x\|^2 S^*$$

$$A_y^T A_y = S\, \|D_y\|^2 S^*$$

$$I + \lambda(A_x^T A_x + A_y^T A_y) = I + \lambda S(\|D_x\|^2 + \|D_y\|^2)S^*$$

$$= S\, \left(I + \lambda\left(\|D_x\|^2 + \|D_y\|^2\right)\right)\, S^*$$

IMAGE RESTORATION

$$\hat{X} = \left(I + \lambda(A_x^T A_x + A_y^T A_y)\right)^{-1} \; Y$$

$$I + \lambda(A_x^T A_x + A_y^T A_y) = S \; \left(I + \lambda\left(\|D_x\|^2 + \|D_y\|^2\right)\right) \; S^*$$

$$\left(I + \lambda(A_x^T A_x + A_y^T A_y)\right)^{-1} = S\left(I + \lambda\left(\|D_x\|^2 + \|D_y\|^2\right)\right)^{-1} S^*$$

$$\hat{X} = S\left(I + \lambda\left(\|D_x\|^2 + \|D_y\|^2\right)\right)^{-1} S^* Y$$

```
Yf = fft2(Y)
Gxf, Gyf = fft2(Gx), fft2(Gy)
ratio = 1 + lambda*(np.abs(Gxf)**2 + np.abs(Gyf)**2)
Xf = Yf / ratio
X = ifft2(Xf)
```

What is this doing ?

It's down-weighting frequency components by a (real) factor where $\|D_x\|^2$ and $\|D_y\|^2$ are high.

Those are high for higher frequencies, because G_x and G_y are derivative filters.

So this operation down-weights higher frequency components $\Rightarrow$ Smooths the image.

IMAGE RESTORATION

Denoising

$$\hat{X} = \arg\min_X \sum_n \|X[n] - Y[n]\|^2 + \lambda \sum_n \left(((G_x * X)[n])^2 + ((G_y * X)[n])^2 \right)$$

De-blurring

Say we know that our image has been blurred by a *known* blur kernel k

$$Y[n] = (X * k)[n] + \epsilon[n]$$

$$\hat{X} = \arg\min_X \sum_n \|(X * k)[n] - Y[n]\|^2 + \lambda \sum_n \left(((G_x * X)[n])^2 + ((G_y * X)[n])^2 \right)$$

$$\hat{X} = \arg\min_X \|A_k X - Y\|^2 + \lambda \ \left(\|A_x X\|^2 + \|A_y X\|^2 \right)$$

where A_k represents the action of convolution by blur kernel k.

$$\hat{X} = \left(A_k^T A_k + \lambda (A_x^T A_x + A_y^T A_y) \right)^{-1} \ A_k^T Y$$

Note that there is now $A_k^T Y$ instead of just Y.

IMAGE RESTORATION

De-blurring / De-convolution

$$\hat{X} = \arg\min_X \|A_k X - Y\|^2 + \lambda \left(\|A_x X\|^2 + \|A_y X\|^2\right)$$
$$= \left(A_k^T A_k + \lambda(A_x^T A_x + A_y^T A_y)\right)^{-1} A_k^T Y$$

We can do this in the Fourier domain again (assuming A_k represents circular convolution).

$$\hat{X} = S\left(\|D_k\|^2 + \lambda\left(\|D_x\|^2 + \|D_y\|^2\right)\right)^{-1} S^* A_k^T Y$$

$$\hat{X} = S\left(\|D_k\|^2 + \lambda\left(\|D_x\|^2 + \|D_y\|^2\right)\right)^{-1} S^* \, (S D_k^* S^*) Y$$

$$\hat{X} = S\left(\|D_k\|^2 + \lambda\left(\|D_x\|^2 + \|D_y\|^2\right)\right)^{-1} D_k^* \, S^* Y$$

$$X_f[u,v] = \frac{K_f[u,v]^*}{\|K_f[u,v]\|^2 + \lambda\left(\|G_{xf}[u,v]\|^2 + \|G_{yf}[u,v]\|^2\right)} Y_f[u,v]$$

If $\lambda = 0$, this reduces to:

$$X_f[u,v] = \frac{K_f[u,v]^*}{\|K_f[u,v]\|^2} Y_f[u,v] = \frac{Y_f[u,v]}{K_f[u,v]}, \text{ as } \|K_f\|^2 = K_f K_f^*$$

IMAGE RESTORATION

De-blurring / De-convolution

$$\begin{aligned}\hat{X} &= \arg\min_X \|A_k X - Y\|^2 + \lambda \left(\|A_x X\|^2 + \|A_y X\|^2\right) \\ &= \left(A_k^T A_k + \lambda(A_x^T A_x + A_y^T A_y)\right)^{-1} A_k^T Y\end{aligned}$$

$$X_f[u,v] = \frac{K_f[u,v]^*}{\|K_f[u,v]\|^2 + \lambda\left(\|G_{xf}[u,v]\|^2 + \|G_{yf}[u,v]\|^2\right)} Y_f[u,v]$$

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But $K_f[u,v]$ may be zero or close to zero, in which case dividing will amplify noise.

So the Fourier transform of the kernel k is telling us which frequency components are severely attenuated by the kernel.

The "regularization" with $\lambda > 0$ helps stabilize the inversion, because if $K_f[u,v]$ is low for some (u,v), then the factor will downscale the input coefficient $Y[u,v]$.

This is called Wiener filtering or Wiener Deconvolution.

IMAGE RESTORATION

De-blurring / De-convolution

$$\begin{aligned}\hat{X} &= \arg\min_X \|A_k X - Y\|^2 + \lambda \left(\|A_x X\|^2 + \|A_y X\|^2\right) \\ &= \left(A_k^T A_k + \lambda(A_x^T A_x + A_y^T A_y)\right)^{-1} A_k^T Y\end{aligned}$$

$$X_f[u,v] = \frac{K_f[u,v]^*}{\|K_f[u,v]\|^2 + \lambda\left(\|G_{xf}[u,v]\|^2 + \|G_{yf}[u,v]\|^2\right)} Y_f[u,v]$$

If you're going to use this method to do de-convolution, you will have to account for the fact that your observed image was a "valid" convolution, and not a 'circular' convolution.

A good approximation is to do what's called "edge tapering". First pad the image Y to a bigger image, where you make values go down smoothly to 0.

Then do the deconvolution, and crop out the central part.

IMAGE RESTORATION

We discussed cases when you know of a basis (wavelet / Fourier) where you can diagonalize your quadratic system matrix, and have a closed form expression for its inverse.

Not always the case. What if you wanted to exactly model valid convolution (not approximate it as circular) ? What if you observed values at a subset of pixels ?

Generally, what if you wanted to compute $X = Q^{-1}Y$ for some arbitrary symmetric positive-definite Q.

IMAGE RESTORATION

$$X = Q^{-1} Y$$

Let's consider a case where you can form Q.

- Never compute Q^{-1}, and then multiply by Y.
  - Numerically unstable, more expensive.
- Call `scipy.linalg.solve`:
  - Cholesky / LDL Decomposition: $Q = L\,D\,L^T$
  - Always exists for a positive definite matrix. L is lower triangular.
  - Solve $Qx = b \rightarrow LDL^T x = b \rightarrow Ly = b, L^T x = D^{-1} y$

$$\begin{bmatrix} a & 0 & 0 & 0 & \dots \\ q & c & 0 & 0 & \dots \\ d & e & f & 0 & \dots \\ & & \vdots & & \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \end{bmatrix}$$

IMAGE RESTORATION

$$X = Q^{-1}Y$$

More generally, when $Q = A_1^T A_1 + A_2^T A_2 + \ldots$, where $A_1, A_2, \ldots$ are sparse operations, that involve convolutions and element-wise operations.

- If A_1 is convolution with k, then you can get the effect of multiplying $A_1^T A_1$ with an image Y by
  - Convolving with k first
  - Then, convolving the result with the flipped version of k
- If A_1 is valid convolution, A_1^T will correspond to "full" convolution with flipped version of k.
- If A_2 is convolution with k followed by elmement-wise multiplication with a mask image, then $A_2^T A_2$ is
  - Convolution with k
  - Multiply by mask
  - Multiply by mask again
  - Convolution with flipped version of k
- So even when we can't form Q, we can carry out the actions QY, as well as $Z^T QY$
  - Compute QY
  - Take element-wise product of the result with Z and sum.

IMAGE RESTORATION

$$x = Q^{-1}b$$

Solve by the Conjugate Gradient method.

- Generic algorithm for solving $Qx = b$ for symmetric positive definite Q.
- Useful when you can multiply by Q but not 'form' it.

Basic Idea

- For a given set of vectors $\{p_1, p_2, \ldots p_N\}$
  - that are same size as x
  - linearly independent
  - N = dimensionality of x
- We can write any $x = \sum_i \alpha_i p_i$
- If we also choose the vectors to be 'conjugate' such that $p_i^T Q p_j = 0$ for $i \neq j$:

$$Qx = b \rightarrow p_k^T Qx = p_k^T b \rightarrow \alpha_i p_k^T Q p_k = p_k^T b \rightarrow \alpha_i = \frac{p_k^T b}{p_k^T Q p_k}$$

CONJUGATE GRADIENT

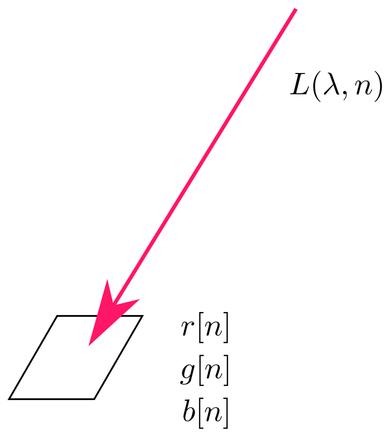
Iterative Algorithm

- Begin with some guess x_0 for x (say all zeros)
- $k = 0, r_0 \leftarrow b - Qx_0, \quad p_0 \leftarrow r_0$
- Repeat
  - $\alpha_k \leftarrow \frac{r_k^T r_k}{p_k^T Q p_k}$
  - $x_{k+1} = x_k + \alpha_k p_k$
  - $r_{k+1} = r_k - \alpha_k Q p_k$
  - $\beta_k = \frac{r_{k+1}^T r_{k+1}}{r_k^T r_k}$
  - $p_{k+1} = r_{k+1} + \beta_k p_k$
  - $k = k + 1$

Stop at some measure of convergence. Pre-conditioned variants. Additional reading: https://en.wikipedia.org/wiki/Conjugate_gradient_method

Think about what you would do when: $Q = \left(A_k^T A_k + \lambda(A_x^T A_x + A_y^T A_y)\right), \quad b = A_k^T Y$

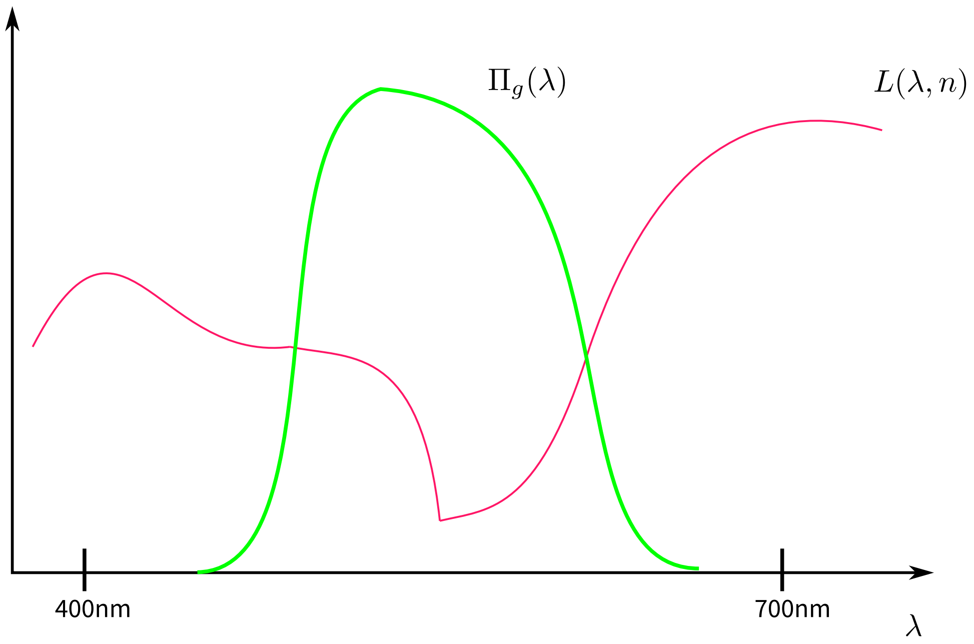
COLOR



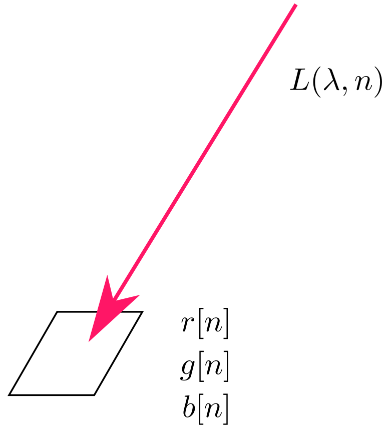
$$r[n] = \int L(\lambda, n)\Pi_r(\lambda)d\lambda$$

$$\boxed{g[n] = \int L(\lambda, n)\Pi_g(\lambda)d\lambda}$$

$$b[n] = \int L(\lambda, n)\Pi_b(\lambda)d\lambda$$



COLOR



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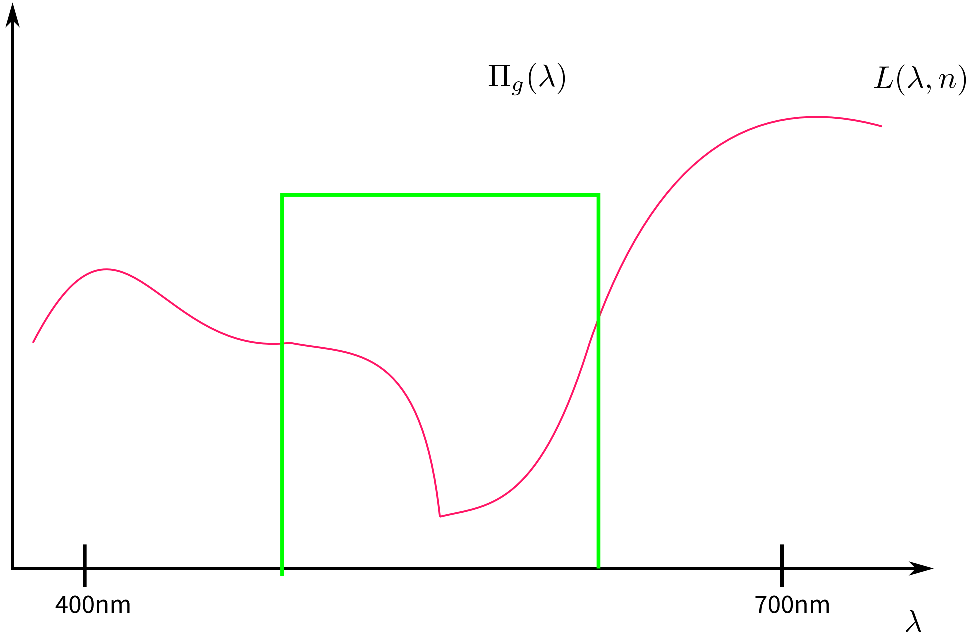
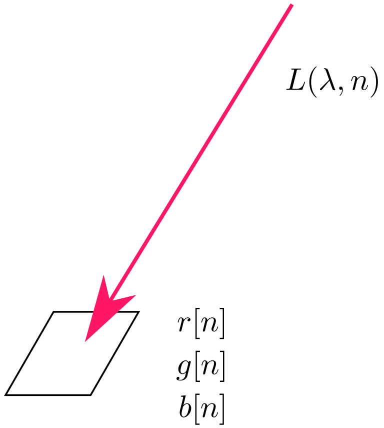


Simple View:
Total / Average Intensity in "Green Part" of the spectrum.

COLOR

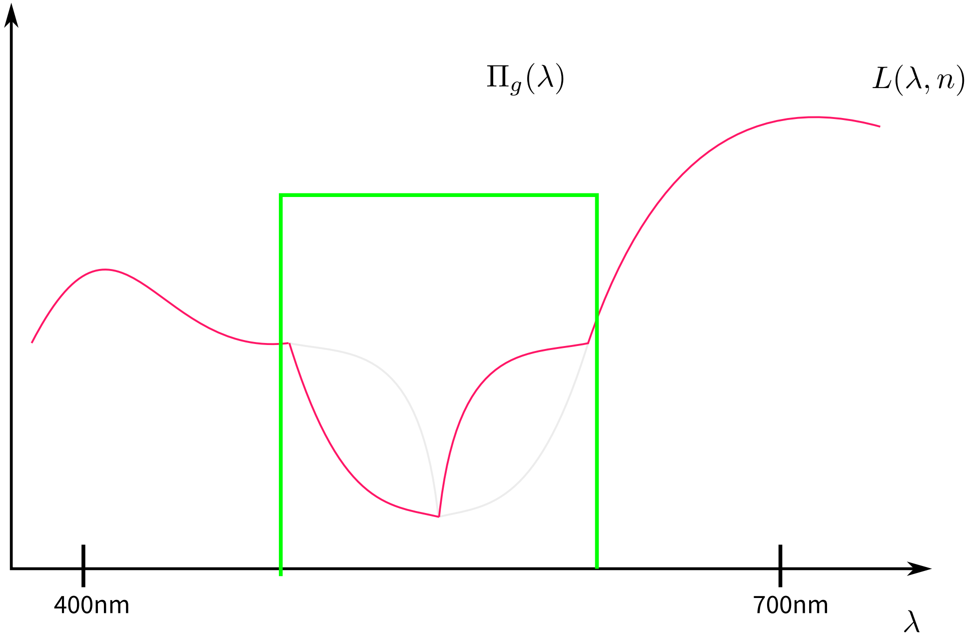


$$r[n] = \int L(\lambda, n)\Pi_r(\lambda)d\lambda$$

$$\boxed{g[n] = \int L(\lambda, n)\Pi_g(\lambda)d\lambda}$$

$$b[n] = \int L(\lambda, n)\Pi_b(\lambda)d\lambda$$

Metamers: Different L that have the same measured RGB values.

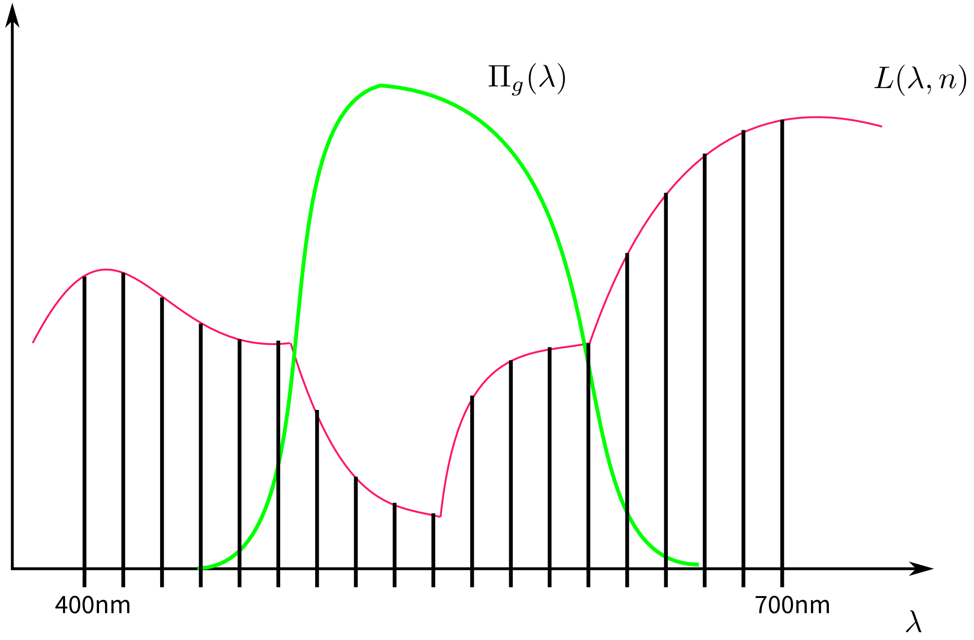


Simple View:
Total / Average Intensity in "Green Part" of the spectrum.

COLOR

For simplicity,
 Have discrete wavelengths
 Approximate integration as summation

$$r[n] = \int L(\lambda, n)\Pi_r(\lambda)d\lambda$$
$$g[n] = \int L(\lambda, n)\Pi_g(\lambda)d\lambda$$
$$b[n] = \int L(\lambda, n)\Pi_b(\lambda)d\lambda$$



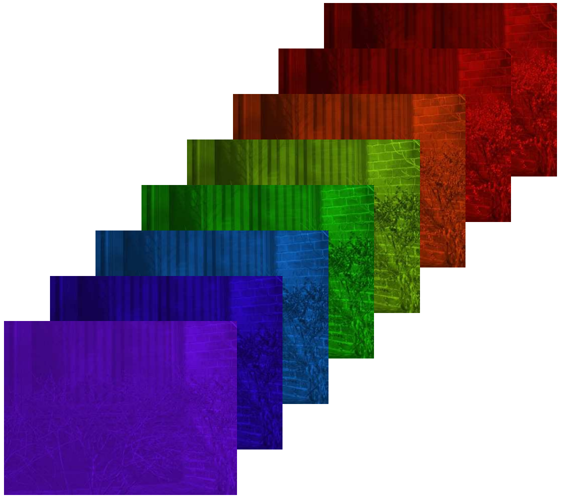
$$L(\lambda, n) \rightarrow L[\lambda, n] \text{ or } L[n] \in \mathbb{R}^B$$

COLOR

$$r[n] = \langle L[n], \Pi_r \rangle$$

$$g[n] = \langle L[n], \Pi_g \rangle$$

$$b[n] = \langle L[n], \Pi_b \rangle$$



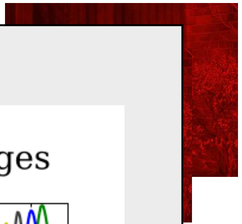
Think of the incident light being a B (>> 3) channel image $L[n]$

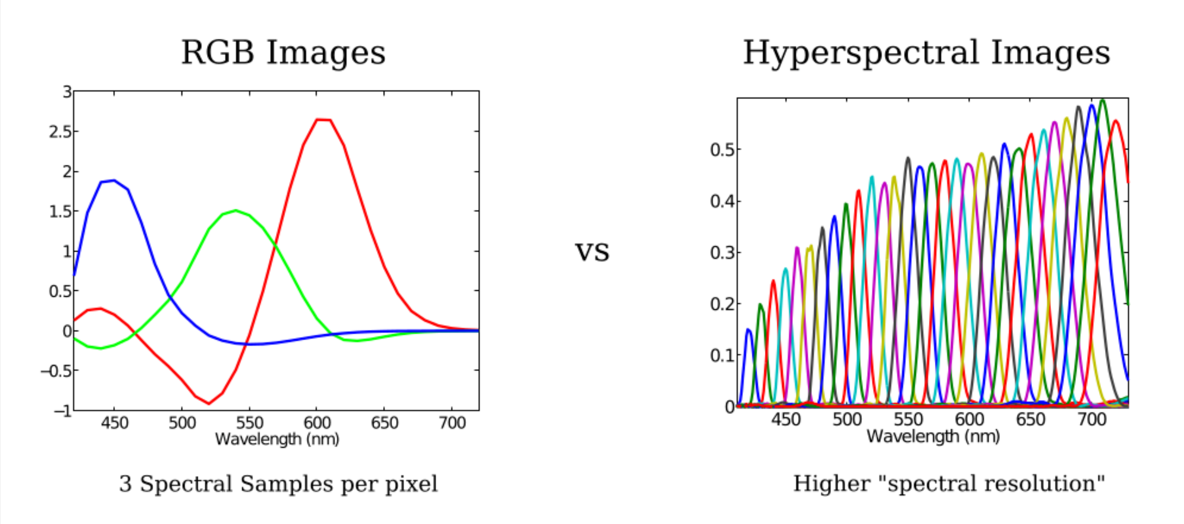
$$L(\lambda, n) \rightarrow L[\lambda, n] \text{ or } L[n] \in \mathbb{R}^B$$

COLOR

$X[n]$

I





There are cameras that actually capture such "hyperspectral" images.

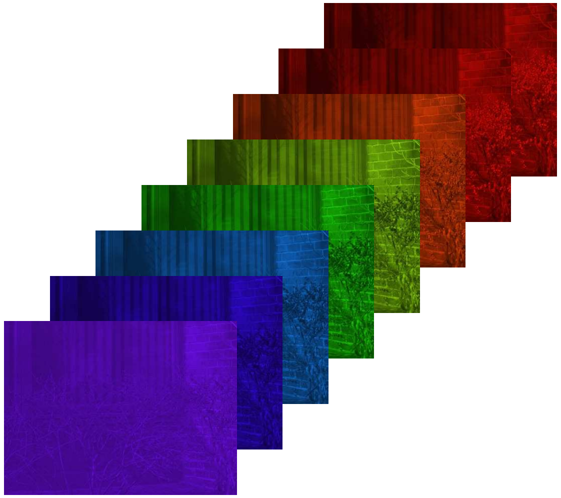
$L(\lambda, n) \rightarrow L[\lambda, n]$ or $L[n] \in \mathbb{R}^B$

COLOR

$$X[n] = \Pi^T \; L[n],$$

$$\Pi = [\Pi_r \quad \Pi_g \quad \Pi_b]$$

(B x 3 Matrix)



Think of the incident light being a B (>> 3) channel image $L[n]$

- 3 Dimensional Projection from higher dimensional space

- Invariant to changes in the "null space" of Π