CSE 559A: Computer Vision

Fall 2018: T-R: 11:30-1pm @ Lopata 101

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http://www.cse.wustl.edu/~ayan/courses/cse559a/

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GENERAL

- Look at Proposal Feedback
- **Important**: This Friday, Office Hours will be shorter.
  - Only from 10:30AM - 11 AM (Lopata 103)
  - Recitation Next Friday
- Colloquium of Potential Interest
  - "Visualizing Scalar Data with Computational Topology and Machine Learning" - Josh Levine from UA
  - 11 AM - Noon, Friday (Lopata 101)
- Advertisement: New Course being offered next semester
  - CSE 659A: Advances in Computer Vision

MACHINE LEARNING

$$w = \arg\min_w \frac{1}{T} \sum_t C_t(w)$$

$$C_t(w) = y_t \log\left[1 + \exp(-w^T \tilde{x}_t)\right] + (1 - y_t) \log\left[1 + \exp(w^T \tilde{x}_t)\right]$$

- Defined linear classifier on augmented vector $\tilde{x}$
- Used gradient descent to learn w.
  - Looked at behavior of gradients.
  - Simplified computation with stochasticity.
- At test time, sign of $w^T \tilde{x}$ gives us our label.

This is for binary classification. What about the multi-class case ? $y \in \{1, 2, 3, \ldots C\}$

MACHINE LEARNING

Multi-Class Classification

- Want to map an input x to a class label $y \in \{1, 2, 3, \ldots C\}$
- Binary case: f outputs a single number between 0,1 that represents $P(y = 1)$.
- Multi-class case: f outputs a C dimensional vector that represents a probability distribution over C classes.

$$f(x; W) = \text{SoftMax}(W^T \tilde{x}) = [p_1, p_2, p_3, \ldots p_C]^T$$

- Here our learnable parameter is now the $N \times C$ matrix W (N is length of feature vector $\tilde{x}$).
- p_i represents the probability of class i
- Each $p_i > 0$, and $\sum_i p_i = 1$
- SoftMax is a generalization of Sigmoid

$$[p_1, p_2, \ldots]^T = \text{SoftMax}([l_1, l_2, \ldots]^T) \rightarrow p_i = \frac{\exp(l_i)}{\sum_{i'} \exp(l_{i'})}$$

- At Test Time: $y = \arg\max_i p_i$
- $y = \arg\max_i l_i$

MACHINE LEARNING

Multi-Class Classification

$$f(x;W) = \mathrm{SoftMax}(W^T\tilde{x}) = [p_1, p_2, p_3, \ldots p_C]^T$$
$$[p_1, p_2, \ldots]^T = \mathrm{SoftMax}([l_1, l_2, \ldots]^T) \rightarrow p_i = \frac{\exp(l_i)}{\sum_{i'} \exp(l_{i'})}$$

What about the Loss ?

Multi-Class Cross Entropy Loss

$$L(y, f(x)) = L(y, [p_1, p_2, \ldots]^T) = -\log p_y$$

- Another way to write it:
  - $y^1 = [\delta_1, \delta_2, \ldots]$, where $\delta_i = 1$ if $y = i$ and 0 otherwise.
  - Called a 1-Hot encoding of the class
  - y^1 also represents a "probability distribution", where the right class has probability 1.
  - In some cases, if you have uncertainty in your training data, y^1 could be a distribution too.

$$L(y^1 = [\delta_1, \delta_2, \ldots], [p_1, p_2, \ldots]^T) = -\sum_i \delta_i \log p_i$$

MACHINE LEARNING

Multi-Class Classification

$$[l_1, l_2, \ldots]^T = W^T\tilde{x}$$
$$p_i = \frac{\exp(l_i)}{\sum_{i'} \exp(l_{i'})}$$
$$L([\delta_1, \delta_2, \ldots], [p_1, p_2, \ldots]^T) = -\sum_i \delta_i \log p_i$$

- We're going to use gradient descent to learn W. What is $\nabla_W L$?
- First, what is $\frac{\partial L}{\partial l_i}$? Take 5 mins.
- Derivative is $p_i - \delta_i$
  - This means that you'll get gradients for all classes (not just the true class)
  - Negative gradient wants you to increase probability for right class, and decrease for other classes
- What is $\nabla_W L$? Take a few minutes!

$$\nabla_W L = \tilde{x} \quad [p_1 - \delta_1, p_2 - \delta_2, \ldots]$$

This is a matrix multiply or outer-product of an $N \times 1$ vector with an $1 \times C$ vector.

MACHINE LEARNING

- For regression and both binary and multi-class classification:
- Defined linear classifier on augmented vector $\tilde{x}$
- Run optimization to learn parameters

The problem is:

- The definition of augmented vector $\tilde{x}$ is hand-crafted
- We have manually engineered our features.
- The only thing we're learning is a linear classifier on top.

Want to learn the features themselves !

Given that SGD works, what's stopping us from learning a function g such that $g(x) = \tilde{x}$?

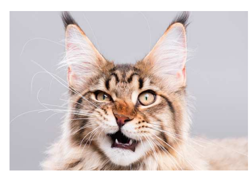
CLASSIFICATION

$$x \xrightarrow{\text{Encode}} \tilde{x} \xrightarrow{\text{Learn } w} w^T\tilde{x} \xrightarrow{\text{Classify}}$$

> 0 True
< 0 False

Hand crafted

Automatic
Data-driven



$\tilde{x}$?

Cat or not Cat ?

- What is an encoding such that a 'linear' classifier on it will suffice ?
- Just list of pixels / quadratic (now N2 dimensional vector) ?

- Kernel methods help with dimensionalit, but still hand-crafted.

CLASSIFICATION

- Learn $\tilde{x} = g(x;\theta)$

$$w = \arg\min_{w} \frac{1}{T} \sum_t y_t \log\left[1 + \exp(-w^T \tilde{x}_t)\right] + (1 - y_t) \log\left[1 + \exp(w^T \tilde{x}_t)\right]$$

$$\theta, w = \arg\min_{\theta,w} \frac{1}{T} \sum_t y_t \log\left[1 + \exp(-w^T g(x_t;\theta))\right] + (1 - y_t) \log\left[1 + \exp(w^T g(x_t;\theta))\right]$$

- Again, use (stochastic) gradient descent.
  - But this time, the cost is no longer convex.
  - Turns out .. doesn't matter (sort of).

Recall in the previous case: (where C_t is the cost of one sample)

$$\nabla_w C_t = \tilde{x}_t \quad \left[\frac{\exp(w^T \tilde{x}_t)}{1 + \exp(w^T \tilde{x}_t)} - y_t\right]$$

What about now ?

Exactly the same, with $\tilde{x} = g(x;\theta)$ for the current value of θ.

CLASSIFICATION

- Learn $\tilde{x} = g(x;\theta)$

$$\theta, w = \arg\min_{\theta,w} \frac{1}{T} \sum_t y_t \log\left[1 + \exp(-w^T g(x_t;\theta))\right] + (1 - y_t) \log\left[1 + \exp(w^T g(x_t;\theta))\right]$$

$$\nabla_w C_t = \tilde{x}_t \quad \left[\frac{\exp(w^T \tilde{x}_t)}{1 + \exp(w^T \tilde{x}_t)} - y_t\right]$$

What about $\nabla_\theta C_t$?

First, what is the $\nabla_{\tilde{x}_t} C_t$?

$$\nabla_{\tilde{x}_t} C_t = \quad w \quad \left[\frac{\exp(w^T \tilde{x}_t)}{1 + \exp(w^T \tilde{x}_t)} - y_t\right]$$

CLASSIFICATION

- Learn $\tilde{x} = g(x;\theta)$

$$\theta, w = \arg\min_{\theta,w} \frac{1}{T} \sum_t y_t \log\left[1 + \exp(-w^T g(x_t;\theta))\right] + (1 - y_t) \log\left[1 + \exp(w^T g(x_t;\theta))\right]$$

$$\nabla_{\tilde{x}_t} C_t = \quad w \quad \left[\frac{\exp(w^T \tilde{x}_t)}{1 + \exp(w^T \tilde{x}_t)} - y_t\right]$$

- Now, let's say θ was an $M \times N$ matrix, and $g(x;\theta) = \theta\, x$.
  - N is the length of the vector x
  - M is the length of the encoded vector $\tilde{x}$

What is $\nabla_\theta C_t$?

$$\nabla_\theta C_t = \left(\nabla_{\tilde{x}_t} C_t\right) x_t^T$$

- This is actually a linear classifier on x
  - $w^T \theta\, x = (\theta^T w)^T\, x = \tilde{w}^T\, x$
- But because of our factorization, is no longer convex.
- If we want to increase the expressive power of our classifier, g has to be non-linear !

CLASSIFICATION

The Multi-Layer Perceptron

$$x \xrightarrow{h = \theta x} h \xrightarrow{\tilde{h}^j = \kappa(h^j)} \tilde{h} \xrightarrow{y = w^T \tilde{h}} y \xrightarrow{p = \sigma(y)} p$$

- κ is an "element-wise" non-linearity.
  - For example $\kappa(x) = \sigma(x)$. More on this later.
  - Has no learnable parameters.
- σ is our sigmoid to convert log-odds to probability.

$$\sigma(y) = \frac{\exp(y)}{1 + \exp(y)}$$

- Multiplication by θ and action of κ is a "layer".
  - Called a "hidden" layer, because you're learning a "latent representation".
  - Don't have direct access to the true value of its outputs
  - Learning a representation that jointly with a learned classifier is optimal

CLASSIFICATION

The Multi-Layer Perceptron

$$x \xrightarrow{h=\theta x} h \xrightarrow{\tilde{h}^j=\kappa(h^j)} \tilde{h} \xrightarrow{y=w^T\tilde{h}} y \xrightarrow{p=\sigma(y)} p$$

- This is a neural network:
  - A complex function formed by *composition* of "simple" linear and non-linear functions.
- This network has learnable parameters θ, w.
- Train by gradient descent with respect to classification loss.
- What are the gradients ?

Doing this manually is going to get old really fast.

Autograd

- Express complex function as a *composition* of simpler functions.
- Store this as nodes in a 'computation graph'
- Use chain rule to automatically back-propagate

Popular Autograd Systems: Tensorflow, Torch, Caffe, MXNet, Theano, ...

We'll write our own!

AUTOGRAD / BACK-PROPAGATION

- Say we want to minimize a loss L, that is a function of parameters and training data.
- Let's say for a parameter θ we can write:
  $$L = f(x); x = g(\theta, y)$$
  where y is independent of θ, and f does not use θ except through x.
- Now, let's say I gave you the value of y and the gradient of L with respect to x.
  - x is an $N-$ dimensional vector
  - θ is an $M-$ dimensional vector (if its a matrix, just think of each element as a separate paramter)

Express $\frac{\partial L}{\partial \theta^j}$ in terms of $\frac{\partial L}{\partial x^i}$ and $\frac{\partial g(\theta,y)^i}{\partial \theta^j}$: which is the partial derivative of one of the dimensions of the outputs of g with respect to one of the dimensions of its inputs.

For every j

$$\frac{\partial L}{\partial \theta^j} = \sum_i \frac{\partial L}{\partial x^i} \frac{\partial g(\theta, y)^i}{\partial \theta^j}$$

We can similarly compute gradients for the "other" input to g, i.e. y.

AUTOGRAD / BACK-PROPAGATION

$$L = f(x, x'); x = g(\theta, y), x' = g'(\theta, y')$$

Let's say a specific variable had two "paths" to the loss.

$$\frac{\partial L}{\partial \theta^j} = \sum_i \frac{\partial L}{\partial x^i} \frac{\partial g(\theta, y)^i}{\partial \theta^j} + \sum_i \frac{\partial L}{\partial x'^i} \frac{\partial g'(\theta, y')^i}{\partial \theta^j}$$

AUTOGRAD / BACK-PROPAGATION

Our very own autograd system:

- Build a directed computation graph with a (python) list of nodes
  G = [n1,n2,n3 ...]
- Each node is an "object" that is one of three kinds:
  - Input
  - Parameter
  - Operation . . .

We will define the graph by calling functions that define functional relationships.

```
import edf

x = edf.Input()
theta = edf.Parameter()

y = edf.matmul(theta,x)
y = edf.tanh(y)

w = edf.Parameter()
y = edf.matmul(w,y)
```

AUTOGRAD / BACK-PROPAGATION

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```

- Each of these statements adds a node to the list of nodes.
- Operation nodes are added by matmul, tanh, etc., and are linked to previous nodes that appear before it in the list as input.
- Every node object is going to have a member element n.top which will be the value of its "output"
  - This can be an arbitrary shaped array.
- For input and parameter nodes, these top values will just be set (or updated by SGD).
- For operation nodes, the top values will be computed from the top values of their inputs.
  - Every operation node will be an object of a class that has a function called forward.
- A forward pass will begin with values of all inputs and parameters set.
- Then we will go through the list of nodes in order, and compute the value of all operation nodes.

AUTOGRAD / BACK-PROPAGATION

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y = edf.matmul(w,y)
```

- A forward pass will begin with values of all inputs and parameters set.
- Then we will go through the list of nodes in order, and compute the value of all operation nodes.
- Because nodes were added in order, if we go through them in order, the tops of our inputs will be available.

AUTOGRAD / BACK-PROPAGATION

```
import edf

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theta = edf.Parameter()

y = edf.matmul(theta,x)
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w = edf.Parameter()
y = edf.matmul(w,y)
```

Somewhere in the training loop, where the values of parameters have been set before.

```
x.set(...)
edf.Forward()
print(y.top)
```

- And this will give us the value of the output.
- But now, we want to compute "gradients".
- For each "operation" class, we will also define a function `backward`.
- All operation and paramter nodes will also have an element called `grad`.
- We will have to then back-propagate gradients in order.