CSE 513T Theory of Artificial Intelligence & Machine Learning March 26, 2015

Lecture 20

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Overview

1. Partial Models

2. Quotient Operator

3. Well-Founded Semantics: existence and algorithm

1 From Last Time

Negation-as-failure -"not ℓ"

1.1 intended meaning

$[\ell_1 \wedge \ell_2 \wedge \ldots \wedge \ell_r \wedge \text{ not } \ell_{r+1} \wedge \ldots \wedge \text{ not } \ell_{r+s}] \Rightarrow \ell$
It means "if ℓ_1 and ℓ_2 and ... ℓ_r has been proved and ℓ_{r+1} and ... ℓ_{r+s} cannot be proven, then we may infer ℓ"
e.g. default reasoning
[bird $\wedge$ not ($\neg$Fly)] $\Rightarrow$ Fly
"if this is a bird and I don't have that it can't fly, then it can fly."

1.2 Troubling cases

1.[not x_1] $\Rightarrow x_2$
2.[not x_2] $\Rightarrow x_1$ } 1 and 2 are two, mutually inconsistent (internally consistent) valuations
3.[not x_3] $\Rightarrow x_3$ } circular definition
4.[x_1] $\Rightarrow x_3$ } suddenly a unique, consistent valuation

2 Partial Models

We'll weaken the interpretation of these "not ℓ" to sometimes avoid committing to whether or not ℓ can be proven.
Eg. I don't want to commit to <u>any</u> of x_1, x_2 or x_3 being provable in the KB 1-3

Definition 1 *A* partial model *is given by disjoint sets of literals ($L_\top$, $L_\bot$) such that for every rule in the KB, [$\ell_1 \wedge \ldots \wedge \ell_r \wedge$ not $\ell_1' \wedge \ldots \wedge$ not ℓ_s'] $\Rightarrow \ell$*
1) if all of ℓ_1, ℓ_2, ..., ℓ_r, not ℓ_1', ..., not $\ell_s' \in L_\top$, then $\ell \in L_\top$
2) if $\ell \in L_\bot$, then at least one of ℓ_1, ℓ_2, ..., ℓ_r, not ℓ_1', ..., not ℓ_s' is in $L_\bot$ also.

We treat facts as rules with empty bodies; hence, [ ] $\Rightarrow \ell$ requires $\ell \in L_\top$.

For example, for any KB, taking $L_\top$ to be the set of literals provable by chaining on KB (treating the 'not ℓ' as new symbols), and taking $L_\bot = \emptyset$ is always a partial model of KB.

A more informative partial model is

Definition 2 *A least* partial model *in a set of partial models M is $(L_\top{}^*, L_\bot{}^*) \in M$ such that for any other $(L_\top, L_\bot) \in M$, $L_\top \supseteq L_\top{}^*$ and $L_\bot \subseteq L_\bot{}^*$.*
(i.e, asserts that the fewest literals are provable/most literals are unprovable)

Theorem 2.1 *The partial model obtained by taking $L_\top{}^*$ to be the set of all literals with chaining proofs in KB (treating the 'not ℓ' as new literals) and taking $L_\bot{}^*$ to be the set of literals that remain unprovable when all of the 'not ℓ' literals are deleted from rule bodies, is the unique least partial model among partial models for which not $\ell \notin L_\top$ (for any not ℓ)*

Proof Whenever a least partial model $(L_\top{}^*, L_\bot{}^*) \in M$ exists, we first note that it must be unique since for any other least partial model $(L_\top{}', L_\bot{}') \in M$, $L_\top{}' \supseteq L_\top{}^* \supseteq L_\top{}'$, so $L_\top{}' = L_\top{}^*$. Likewise, $L_\bot{}' \subseteq L_\bot{}^* \subseteq L_\bot{}'$, so $L_\bot{}' = L_\bot{}^*$, and $(L_\top{}', L_\bot{}') = (L_\top{}^*, L_\bot{}^*)$.
We first show that any partial model $(L_\top, L_\bot) \in M$, $L_\top$ must contain $L_\top{}^*$, by induction on the length of the chaining derivation. A 1-step chaining proof simply asserts a fact of the KB, which is in $L_\top$ by the definition of a partial model; given that all literals provable by a S-1 step chaining proof are in $L_\top$, for any S-step chaining proof, the final step either asserts another fact of KB (in $L_\top$ trivially) or, applies chaining on a rule $[\ell_1 \wedge \ldots \wedge \ell_r] \Rightarrow \ell$ of KB given that $\ell_1, \ldots, \ell_r$ were derived previously (in the first S-1 steps). So, by IH, $\ell_1, \ldots, \ell_r \in L_\top$ and so by (1) in the definition of partial model, $\ell \in L_\top$ as well. Since this is precisely the set of literals in $L_\top{}^*$, $L_\top \supseteq L_\top{}^*$. To see $L_\bot{}^* \supseteq L_\bot$, we note that if $\ell \in L_\bot$ for some such partial model, since no not ℓ' can be in $L_\bot$ by (2) in the definition of partial model, for any rule of KB with ℓ in the head, there must be some other $\ell' \in L_\bot$ (if no such rules exist, $\ell \in L_\bot{}^*$ trivially).
Suppose for contradiction that $\ell \in L_\bot$ but $\ell \notin L_\bot{}^*$, then ℓ was provable by chaining in the KB when all of the 'not ℓ' are deleted. Consider the ℓ with the shortest chaining proof. Then since $\ell \in L_\bot$, by (2) in the definition of partial model, there must be some ℓ' in the body of every rule containing ℓ in the head, such that ℓ' is not a 'not' literal and $\ell' \in L_\bot$. Consider the rule of the KB used to derive ℓ by chaining, we have another $\ell' \in L_\bot$ that cannot be in $L_\bot{}^*$. Since we used ℓ' in this chaining proof of ℓ, we have a shorter chaining proof of ℓ' contradicting the choice of ℓ as the literal with the shortest chaining proof in $L_\bot$-$L_\bot{}^*$. We thus conclude that every $\ell \in L_\bot$ is also in $L_\bot{}^*$.
To see that $L_\bot{}^*$ is as needed for a partial model note that in the modified KB (where we deleted all of the 'not ℓ') each literal ℓ is either provable or not. Since whenever all of the literals in the body of a rule containing ℓ in the head are provable, ℓ is provable itself, by the contra positive, if ℓ is not provable, it cannot be that every literal in the body is provable, hence there is also 'some ℓ' from the body in $L_\bot{}^*$.
■

3 Quotient Operator

We need to incorporate the judgement about the "probability" of literals encoded in a partial model into the negation-as-failure rules of the KB.

Definition 3 *Given a pair$(L_\top, L_\bot)$ of the sets of literals and a KB, the quotient of KB modulo $(L_\top, L_\bot)$, denoted KB/$(L_\top, L_\bot)$, is given by*
(1) substituting 1 for occurrences of not ℓ for any $\ell \in L_\bot$
(2) substituting 0 for occurrences of not ℓ for any $\ell \in L_\top$
(3) simplifying the resulting KB by deleting rules where 0 appears in the body, and deleting occurrences of 1 in the rule bodies.

Example: Consider the KB $\{x_1, [\text{not } x_1] \Rightarrow x_2, [\text{not } x_3] \Rightarrow x_3\}$
The (unique) least partial model is: $L_\top = \{x_1\}$, $L_\bot = \{\neg x_1, \neg x_2, \neg x_3\}$.
KB$/(L_\top, L_\bot) = \{x_1, [\text{not } x_3] \Rightarrow x_3\}$
its least partial model is $L_\top{}' = \{x_1\}$, $L_\bot{}' = \{x_2, \neg x_1, \neg x_2, \neg x_3\}$
KB$/(L_\top{}', L_\bot{}') = \{x_1, [\text{not } x_3] \Rightarrow x_3\}$
Since we have the same KB, we'll get the same partial model $\rightarrow$ we have converged to a fixed point. This is the interpreted version of the KB in the Well-Founded Semantics.

4 Well-Founded Semantics

Theorem 4.1 *Consider the sequence of pairs of sets of literals given by* $(L_\top^{(0)}, L_\bot^{(0)}) = (\emptyset, \emptyset)$, $(L_\top^{(1)}, L_\bot^{(1)})$ *is the least partial model of* KB$/(L_\top^{(j-1)}, L_\bot^{(i-1)})$ *that does not place any not* ℓ *in* $L_\bot^{(1)}$. *Then the sequence converges to a fixed point* $(L_\top{}^*, L_\bot{}^*)$ *such that for any of the standard literals* ℓ *in KB*
(1) If there is a chaining proof of $\ell \in$ KB$/(L_\top{}^*, L_\bot{}^*)$, *then* $\ell \in L_\top{}^*$
(2) if $\ell \in L_\bot{}^*$, *then there is* <u>no</u> *chaining proof of* ℓ *from* KB$/(L_\top{}^*, L_\bot{}^*)$
and furthermore, there is an algorithm that computes $(L_\top{}^*, L_\bot{}^*)$ *in time* $O(n^2|KB|)$, *where* $|KB|$ *is the size of KB (in bits).*

The setting of not ℓ by $L_\top{}^*$, $L_\bot{}^*$ under the quotient is the Well-Founded Semantics for negation-as-failure in KB.