CSE 347 Analysis of Algorithms February 13, 2019

Lecture 9 Summary

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1. For any s-t cut (A, B), the value of a flow f (from s to t) is equal to the amount of flow leaving A minus the amount of flow returning to A. Therefore, the value of any flow is at most the capacity of any cut (the total capacity of edges crossing from A to B).

2. Ford Fulkerson returns a flow with value equal to the capacity of a cut, and is therefore of maximum possible value (thus also "Max-Flow = Min-Cut")

   (a) F-F terminates when there is no s-t path using edges with positive residual capacity

   (b) If we look at the set of vertices reachable from s using only edges of positive residual capacity A^* when F-F terminates, the edges leaving A^* must be saturated and there must not be any flow returning to A^*.

   (c) Therefore, the value of the flow is equal to the capacity of the corresponding cut.

   (d) Moreover, we can therefore find a minimum cut by finding this set A^* using reachability in the residual graph – also in total time $O(C|E|)$ where C is the total capacity of edges leaving s.

3. We can solve the Maximum Bipartite Matching problem by *reducing* it to (Integer) Maximum Flow, which is solved by F-F.

   (a) A bipartite graph is one that has a vertex set consisting of two disjoint sets L and R such that all edges cross between L and R. A matching is a set of edges that do not share any endpoints.

   (b) Maximum Bipartite Matching: Given a bipartite graph, find a matching of maximum size.

   (c) We will solve maximum bipartite matching using the following algorithm: add a start vertex s with edges to every vertex in L, orient the original edges to point from L to R, and add a vertex t with an edge from every vertex in R. Give all of these edges capacity 1. Now run F-F to obtain an integer maximum flow in this graph. Return the set of edges crossing from L to R with flow 1.

   (d) This algorithm runs in time $O(|V||E|)$.

   (e) Given any matching of size k in the original graph, there is a flow in this graph of value k. Therefore the maximum integer flow has value at least k.

   (f) Next time, we will show that the set of edges returned by our algorithm is a matching with size at least the value of the flow. In turn, this is therefore a matching that is at least as large as any other matching, i.e., a matching of maximum size.