Please follow the collaboration policy.

1. What is the conditional probability that a family with two children has two boys, given that they have at least one boy? Assume that each of the possibilities BB, BG, GB and GG is equally likely, where B represents a boy and G a girl. (5 points)

2. Assume, as in Problem 1, that each of the four ways a family can have two children is equally likely. Are the event E, that a family with two children has two boys, and F, that a family with two children has at least one boy, independent? (5 points)

3. What is the expected value of the sum of the numbers that appear when a pair of fair dice is rolled? (5 points)

4. Given two fair dice, what is the variance of the random variable \( X((i, j)) = 2i \), where \( i \) is the number appearing on the first die and \( j \) is the number appearing on the second die, assuming that the two dice are rolled? (5 points)

5. Average-case complexity of the linear-search algorithm. Consider the linear search problem we discussed in class. (to remind you, the problem was to find the location of a particular number \( x \) in an unsorted array of distinct numbers; the search terminated when \( x \) was found in the array, or return -1 if \( x \) was not in the array.) Assume that the probability that \( x \) is one of the \( n \) numbers is \( p \) and it is equally likely that \( x \) is any of the \( n \) numbers in the array. What is the average complexity, which is measured by the average number of comparisons executed? (hint: there are \( n+1 \) possible types of input: \( x \) can be any of the \( n \) numbers in the list and \( x \) is not in the list, which we treat as a single input.) (10 points)

6. One type of mortgage is call variable mortgage, in which the monthly interest that the lender collects is at a low rate for a certain number of months initially, and then the interest will be locked into whatever the rate the central bank sets based on the financial market conditions. Assume that a borrower got a 30-year variable loan (meaning both the loan and interest need to be paid back within 30 years) of $100,000 at an initial monthly interest rate of 5% for the first 2 years, and then the monthly interest rate was locked into 6%. What is the total amount the borrower has to pay in total if the load was exactly paid back in 30 years? (8 points)

7. A young pair of rabbits (one of each sex) is placed on an island. A pair of rabbits does not breed until they are 2 months old. After they are 2 months old, each pair of rabbits produces another pair each month. Find a recurrence relation for the number of pairs of rabbits on the island after \( n \) months, assuming that no rabbits ever die. (10 points)

8. Find a recurrence relation and give initial conditions for the number of bit strings of length \( n \) that do not have two consecutive 0s. How many such bit strings are there of lengths 5, 6 and 7? (10 points)

9. (extra 10 points) Prove the equation, \( k \times C(n, k) = n \times C(n-1, k-1) \), using a) a combinatorial argument and b) meth derivation, where \( k \) is less than or equal \( n \).