Please follow the collaboration policy.

1. Each user of a computer system has a password, which is five to seven characters long, where each character is an uppercase or lowercase letter or a digit. Each password must contain at least two digits which cannot be next to each other, i.e., no two digits can be next to one another. For example, 2Bob3 is valid, while 2Bob32 is not. How many possible passwords are there? (10 points)

2. What is the least number of area codes needed to guarantee that the 25 million phones in a state have distinct 10-digit telephone numbers? (Assume that telephone numbers are of the form NXX-NXX-XXXX, where the first three digits form the area code, N represents a digit from 2 to 9 inclusive, and X represents any digit.) (10 points)

3. Assume that in a group of six people, each pair of individuals consists of two friends or two enemies. Show that there are either three mutual friends or three mutual enemies in the group. (15 points)

4. The English alphabet contains 21 consonants and five vowels. How many strings of six lowercase letters of the alphabet contain?
   a. Exactly 1 vowel? (5 points)
   b. Exactly 2 vowels? (5 points)
   c. At least 1 vowel? (5 points)
   d. At least 2 vowels? (5 points)

5. Thirteen people on a softball team show up for a game (3 points each)
   a. How many ways are there to choose 10 player to take the field? (3 points)
   b. How many ways are there to assign the 10 positions by selecting players from the 13 people who show up? (3 points)
   c. Of the 13 people who show up, 3 are women. How many ways are there to choose 10 players to take the field if at least one of these players must be a woman? (3 points)

6. Prove the identity $C(n, r)C(r, k) = C(n, k)C(n-k, r-k)$, where $n$, $r$, and $k$ are nonnegative integers with $r \leq n$ and $k \leq r$ (8 points each)
   a. Using a combinatorial argument.
   b. Using an argument based on definition of combination $C(n, m)$.

7. In the lecture, we discussed a deck of 52 cards. (3 points each)
   a. How many different hands of 5 cards form the deck of 52 are there? (3 points)
   b. What is the probability that a five-card poker hand contains a flush, that is, five cards of the same suit? (3 points)
   c. What is the probability that a five-card poker hand contains a straight flush, that is, five cards of the same suit of consecutive kinds? (3 points)
   d. What is the probability that a poker hand contains a full house, that is, three of one kind and two of another kind. (3 points)

8. Use Mathematical Induction to prove this important theorem on the expansion of powers of binomial expression: (10 points)
   $$(x+y)^n = \sum_{k=0}^{n} C(n, k) x^k y^{n-k}$$

9. A pair of dice is loaded. The probability that a 4 appears o the first die is 2/7, and the probability that a 3 appears on the second die is 2/7. Other outcomes for each die appear with probability 1/7. What is the probability of 7 appearing as the sum of the numbers when the two dice are rolled? (5 points)