A Whirlwind Tour of Game Theory

(Mostly from Fudenberg & Tirole)

Players choose actions, receive rewards based on their own actions and those of the other players.

Example, the Prisoner’s Dilemma:

<table>
<thead>
<tr>
<th></th>
<th>Cooperate</th>
<th>Defect</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cooperate</td>
<td>+3, +3</td>
<td>0, +5</td>
</tr>
<tr>
<td>Defect</td>
<td>+5, 0</td>
<td>+1, +1</td>
</tr>
</tbody>
</table>

Strategies and Nash Equilibrium

A strategy is a specification for how to play the game for a player. A pure strategy defines, for every possible choice a player could make, which action the player picks. A mixed strategy is a probability distribution over strategies.

A Nash equilibrium is a profile of strategies for all players such that each player’s strategy is an optimal response to the other players’ strategies. Formally, a mixed-strategy profile $\sigma^*$ is a Nash equilibrium if for all players $i$:

$$u^i(\sigma^*_i, \sigma^*-i) \geq u^i(s^i, \sigma^*-i) \forall s^i \in S^i$$

Nash equilibrium of Prisoner’s Dilemma: Both players defect!
Matching Pennies

<table>
<thead>
<tr>
<th></th>
<th>H</th>
<th>T</th>
</tr>
</thead>
<tbody>
<tr>
<td>H</td>
<td>+1,−1</td>
<td>−1,+1</td>
</tr>
<tr>
<td>T</td>
<td>−1,+1</td>
<td>+1,−1</td>
</tr>
</tbody>
</table>

No pure strategy equilibria

Nash equilibrium: Both players randomize half and half between actions.

More on Equilibria

Dominated strategies: Strategy $s_i$ (strictly) dominates strategy $s'_i$ if, for all possible strategy combinations of opponents, $s_i$ yields a (strictly) higher payoff than $s'_i$ to player $i$.

Iterated elimination of strictly dominated strategies: Eliminate all strategies which are dominated, relative to opponents’ strategies which have not yet been eliminated.

If iterated elimination of strictly dominated strategies yields a unique strategy $n$-tuple, then this strategy $n$-tuple is the unique Nash equilibrium (and it is strict).

Every Nash equilibrium survives iterated elimination of strictly dominated strategies.
**Multiple Equilibria**

A coordination game:

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>U</td>
<td>9,9</td>
<td>0,8</td>
</tr>
<tr>
<td>D</td>
<td>8,0</td>
<td>7,7</td>
</tr>
</tbody>
</table>

$U, L$ and $D, R$ are both Nash equilibria. What would be reasonable to play? With and without coordination?

While $U, L$ is pareto-dominant, playing $D$ and $R$ are “safer” for the row and column players respectively...

**Existence of Equilibria**

Nash’s theorem, translated: every game with a finite number of actions for each player where each player’s utilities are consistent with the (previously discussed) axioms of utility theory has an equilibrium in mixed strategies.

Idea 1: Reaction correspondences. Player $i$’s reaction correspondence $r_i$ maps each strategy profile $\sigma$ to the set of mixed strategies that maximize player $i$’s payoff when her opponents play $\sigma_{-i}$. Note that $r_i$ depends only on $\sigma_{-i}$, so we don’t really need all of $\sigma$, but it will be useful to think of it this way. Let $r$ be the Cartesian product of all $r_i$. A fixed point of $r$ is a $\sigma$ such that $\sigma \in r(\sigma)$, so that for each player, $\sigma_i \in r_i(\sigma)$. Thus a fixed point of $r$ is a Nash equilibrium.

Kakutani’s FP theorem says that the following are sufficient conditions for $r : \Sigma \to \Sigma$ to have a FP.
1. Σ is a compact, convex, nonempty subset of a finite-dimensional Euclidean space.
   Satisfied, because it's a simplex

2. r(σ) is nonempty for all σ
   Each player's playoffs are linear, and therefore continuous, in her own mixed strategy. Continuous functions on compact sets attain maxima.

3. r(σ) is convex for all σ
   Suppose not. Then ∃σ', σ'' such that λσ' + (1 − λ)σ'' ∉ r(σ) But for each player i,
   \[ u_i(\lambda\sigma_i' + (1 − \lambda)\sigma_i'', \sigma_{-i}) = \lambda u_i(\sigma_i', \sigma_{-i}) + (1 − \lambda)u_i(\sigma_i'', \sigma_{-i}) \]
   so that if both σ' and σ'' are best responses to σ_{-i}, then so is their weighted average.

4. r(·) has a closed graph
   The correspondence r(·) has a closed graph if the graph of r(·) is a closed set. Whenever the sequence (σ^n, ˆσ^n) → (σ, ˆσ), with ˆσ^n ∈ r(σ^n)∀n, then ˆσ ∈ r(σ) (same as upper hemicontinuity)
   Suppose that there is a sequence (σ^n, ˆσ^n) → (σ, ˆσ) such that ˆσ^n ∈ r(σ^n)∀n, but ˆσ ∉ r(σ). Then there exists ε > 0 and σ' such that
   \[ u_i(σ'_i, σ_{-i}) > u_i(ˆσ_i, σ_{-i}) + 3ε \]
   Then, for sufficiently large n,
   \[ u_i(σ'_i, σ^n_{-i}) > u_i(σ'_i, σ_{-i}) − ε > u_i(ˆσ_i, σ_{-i}) + 2ε \]
   which means that σ'_i does strictly better against σ^n_{-i} than ˆσ^n_i does, contradicting our assumption.
Learning in Games∗

How do players reach equilibria?

What if I don’t know what payoffs my opponent will receive?

I can try to learn her actions when we play repeatedly (consider 2-player games for simplicity).

Fictitious play in two player games. Assumes stationarity of opponent’s strategy, and that players do not attempt to influence each others’ future play. Learn weight functions

\[ \kappa_i^t(s^{-i}) = \kappa_{i-1}^t(s^{-i}) + \begin{cases} 1 & \text{if } s_{t-1}^{-i} = s^{-i} \\ 0 & \text{otherwise} \end{cases} \]

Calculate probabilities of the other player playing various moves as:

\[ \gamma_i^t(s^{-i}) = \frac{\kappa_i^t(s^{-i})}{\sum_{\tilde{s}^{-i} \in S^{-i}} \kappa_i^t(\tilde{s}^{-i})} \]

Then choose the best response action.

Fictitious Play (contd.)

If fictitious play converges, it converges to a Nash equilibrium.

If the two players ever play a (strict) NE at time $t$, they will play it thereafter. (Proofs omitted)

If empirical marginal distributions converge, they converge to NE. But this doesn't mean that play is similar!

<table>
<thead>
<tr>
<th>$t$</th>
<th>Player1 Action</th>
<th>Player2 Action</th>
<th>$\kappa_1$</th>
<th>$\kappa_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>T</td>
<td>T</td>
<td>(1.5, 3)</td>
<td>(2, 2.5)</td>
</tr>
<tr>
<td>2</td>
<td>T</td>
<td>H</td>
<td>(2.5, 3)</td>
<td>(2.3, 5)</td>
</tr>
<tr>
<td>3</td>
<td>T</td>
<td>H</td>
<td>(3.5, 3)</td>
<td>(2.4, 5)</td>
</tr>
<tr>
<td>4</td>
<td>H</td>
<td>H</td>
<td>(4.5, 3)</td>
<td>(3.4, 5)</td>
</tr>
<tr>
<td>5</td>
<td>H</td>
<td>H</td>
<td>(5.5, 3)</td>
<td>(4, 4.5)</td>
</tr>
<tr>
<td>6</td>
<td>H</td>
<td>H</td>
<td>(6.5, 3)</td>
<td>(5.4, 5)</td>
</tr>
<tr>
<td>7</td>
<td>H</td>
<td>T</td>
<td>(6.5, 4)</td>
<td>(6.4, 5)</td>
</tr>
</tbody>
</table>

Cycling of actions in fictitious play in the matching pennies game

Universal Consistency

Persistent miscoordination: Players start with weights of $(1, \sqrt{2})$

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>0, 0</td>
<td>1, 1</td>
</tr>
<tr>
<td>B</td>
<td>1, 1</td>
<td>0, 0</td>
</tr>
</tbody>
</table>

A rule $\rho^i$ is said to be $\epsilon$-universally consistent if for any $\rho^{-i}$

$$\lim_{T \to \infty} \sup_{\sigma^i} \max_{\gamma^i} u^i(\sigma^i, \gamma^i) - \frac{1}{T} \sum_{t} u^i(\rho^i_t(h_{t-1})) \leq \epsilon$$

almost surely under the distribution generated by $(\rho^i, \rho^{-i})$, where $h_{t-1}$ is the history up to time $t - 1$, available for the decision-making algorithm at time $t$. 
Back to Experts

Bayesian learning cannot give good payoff guarantees.

- Suppose the true way your opponent’s actions are being generated is not in the support of the prior — want protection from unanticipated play, which can be endogenously determined.

- The Bayesian optimal method guarantees a measure of learning something close to the true model, but provides no guarantees on received utility.

- Can use the notion of experts to bound regret!

Define universal expertise analogously to universal consistency, and bound regret (lost utility) with respect to the best expert, which is a strategy.

The best response function is derived by solving the optimization problem

$$\max_{\mathcal{I}^i} \bar{u}_t^i + \lambda v^i(\mathcal{I}^i)$$

$\bar{u}_t^i$ is the vector of average payoffs player $i$ would receive by using each of the experts

$\mathcal{I}^i$ is a probability distribution over experts

$\lambda$ is a small positive number.

Under technical conditions on $v$, satisfied by the entropy:

$$-\sum_s \sigma(s) \log \sigma(s)$$

we retrieve the exponential weighting scheme, and for every $\epsilon$ there is a $\lambda$ such that our procedure is $\epsilon$-universally expert.