Notes on Generating Random Permutations

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(Adapted partially from Cormen et al’s Introduction to Algorithms.

Problem Statement: We want to generate permutations of 1 ... n uniformly at random, meaning each permutation has probability 1/n! of occurring. Note that this allows us a general means to permute any n elements, say of an array, by permuting the indices of the array.

The first algorithm we consider (Algorithm 1) is obviously correct. Any output is a legal permutation, and the probability of any given permutation is 1/n! because the elements are selected completely at random at each step. The running time is more complicated to analyze. We’ll consider it on average. First let’s consider the question of how many random draws we would expect to have to make in order to generate a new number for some j.

j of the n numbers would be duplicates, so the probability of success (generating a new number) is \( \frac{n-j}{n} \). Then the expectation of the number of trials until the first success is \( \frac{n}{n-j} \) (this is a standard result – when you have Bernoulli trials with probability p of success the expected number of trials until first success is 1/p – look up the geometric distribution if you are interested in the derivation.

So now, for the running time we get:

\[
\sum_{j=0}^{n-1} \frac{n}{n-j} = n \sum_{j=1}^{n} \frac{1}{j} = O(n \log n)
\]

Keep in mind that there is always some chance that the algorithm will not terminate by any specified finite time T.

Input: n
for i in 1 to n do
    used[i] = false
end for
for j in 0 to n - 1 do
    repeat
        temp = randInt(1,n)
        if not(used[temp]) then
            a[j] = temp
            used[temp] = true
        end if
    until a[j] is filled
end for

Algorithm 1: Not so great algorithm for permutation
Input: $n$

for $i$ in 0 to $n - 1$
    $a[i] = i + 1$
end for

for $j$ in 0 to $n - 1$
    swap($a[j]$, $a[randInt(j,n-1)]$)
end for

Algorithm 2: Better algorithm for permutation

Algorithm 2 is a better algorithm. The running time is obvious: $O(n)$. Proving correctness is not so easy. First, define a $k$-permutation of a set of $n$ elements as a sequence containing $k$ of those $n$ elements. There are $\frac{n!}{(n-k)!}$ such permutations (how?).

We now introduce the following loop invariant:
Prior to the $j$th iteration, $a[0\ldots j-1]$ contains each $j$-permutation with probability $\frac{(n-j)!}{n!}$.

Convince yourself that the loop invariant holds for $j = 0, 1$. Now that we have the base case done, we have to show that the loop invariant is maintained at each iteration. Let’s say iteration $j$ ends with: $\langle x_0, x_1, \ldots, x_j \rangle$

Define the following two events
Event $E_1$ is that the first $j - 1$ iterations have yielded $\langle x_0, \ldots, x_{j-1} \rangle$
Event $E_2$ is that the $j$th iteration puts $x_j$ in position $j$

Applying Bayes rule,
$$
\Pr(E_2 \land E_1) = \Pr(E_2|E_1) \Pr(E_1)
$$
$$
= \frac{1}{n-j} \frac{(n-j)!}{n!}
$$
$$
= \frac{(n-j-1)!}{n!}
$$

This proves that the loop invariant is maintained.

Finally, at termination, $j = n$, so $a$ contains each $n$-permutation with probability $\frac{(n-n)!}{n!} = 1/n!$. 