Digital Systems and Information Representation

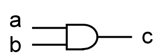
CSE 102

What is Binary?

- Underlying base signals are two-valued:
  - 0 or 1
  - true or false (T or F)
  - high or low (H or L)
- One "bit" is the smallest unambiguous unit of information
- Propositional calculus helps us manipulate (operate on) these base signals

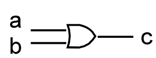
Operations in Propositional Calculus

AND $a \cdot b = c$



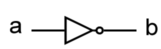
c is true if a is true <u>and</u> b is true

OR $a + b = c$



c is true if a is true <u>or</u> b is true

NOT $a' = b$



b is true if a is false

An Example

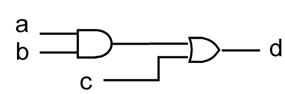
a passed microeconomics course

b passed macroeconomics course

c passed economics survey course

d met economics requirement

$d = a \cdot b + c$



Boolean Algebra

- Boolean algebra (named after 19th century mathematician George Boole) lets us manipulate and reason about expressions of propositional calculus
- Systems based on this algebraic theory are called "digital logic systems"
- All modern computer systems fall in this category

Physical Representation

- Positive logic convention
  - Binary value (1 or 0) is represented by the voltage on a wire (H or L)
  - true, 1 voltage greater than threshold V_H
  - false, 0 voltage less than threshold V_L
  - Voltage gap between V_H and V_L provides safety margin to limit errors

That's Not Enough!

- We are interested in representing signals that have more than just two values
  - numbers
  - text
  - images
  - audio
  - video
  - and much more
- Let's look at a sequence of number systems (unveiling some history in the process)

Start with "counting" numbers

- 1, 2, 3, … (positive integers)
- good at measuring how much stuff I have
  - e.g., 7 cows and 2 goats
- closed under addition and multiplication
  - if a,b are numbers, a + b is too
  - if a,b are numbers, a × b is too

What if I don't have any stuff?

- incorporate 0 into number system
  - I have 0 pigs
- still closed under addition and multiplication
- how do we represent zero in Roman numerals?
  - we don't!
  - the Romans hadn't yet invented zero

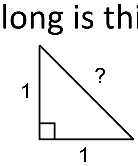
What if I take away more than I have?

- new concept: negative numbers
  - I owe my neighbor 3 chickens, vs.
  - I own -3 chickens
- closed under subtraction as well
- can solve equation x + a = 0 for any integer a

Why can't I divide my stuff 3 ways?

- incorporate rational numbers
  - I have 2 pigs, so when I die…
  - each of my 3 sons gets 2/3 pig
- now closed under division
- can now solve equation ax + b = 0 for arbitrary numbers a,b if a ≠ 0

How long is this line?



- incorporate irrational numbers
- defined "real" numbers
- can now solve equation $x^2 - 2 = 0$
- Note pattern of names:
  - positive to negative
  - rational to irrational

But what about $x^2 + 2 = 0$?

- introduce complex numbers
- vector number system with 2 components:
  - (a, b)   a, b both "real" numbers
- obey following rules:
  - equality: $(a, b) = (c, d)$ iff $a = c$ and $b = d$
  - addition: $(a, b) + (c, d) = (a + c, b + d)$
  - multiplication: $(a, b) \times (c, d) = (ac - bd, ad + bc)$

Interesting fact

- complex numbers with 2nd component equal 0 have same properties as "real" numbers

  $(a, 0) = (c, 0)$ iff $a = c$

  $(a, 0) + (c, 0) = (a + c, 0)$

  $(a, 0) \times (c, 0) = (ac, 0)$

What name to use?

- 1st component of complex number is called "real", why not follow the historical naming conventions?

  ¬ positive ⇒ negative

  ¬ rational ⇒ irrational

  ¬ real ⇒ imaginary
- therefore, 2nd component of complex number came to be called "imaginary"

Back to our equation

- What about the equation we were trying to solve?

  $x^2 + 2 = 0$
- First, rewrite as equation in complex numbers

  $x^2 + (2,0) = (0,0)$
- Second, insert $x = (0, \sqrt{2})$

$$
\begin{aligned}
(0, \sqrt{2})^2 + (2, 0) &= (0, \sqrt{2}) \times (0, \sqrt{2}) + (2, 0) \\
&= (-2, 0) + (2, 0) \\
&= (0, 0) \quad \checkmark
\end{aligned}
$$

Another interesting equation: $x^2 + 1 = 0$

- Initial algebraic manipulation yields:

$$
\begin{aligned}
x^2 + 1 &= 0 \\
x^2 &= -1 \\
x &= \sqrt{-1}
\end{aligned}
$$

- Now try with $x = (0,1)$:

$$
\begin{aligned}
(0, 1)^2 + (1, 0) &= (0, 1) \times (0, 1) + (1, 0) \\
&= (-1, 0) + (1, 0) \\
&= (0, 0) \quad \checkmark
\end{aligned}
$$

- Therefore, $x = (0, 1) = \sqrt{-1}$

New notation for complex numbers

- Let symbol $i = \sqrt{-1}$
- (a, b) can now be written $a + ib$
- since

$$
\begin{aligned}
a + ib &= (a, 0) + (0, 1) \times (b, 0) \\
&= (a, 0) + (0, b) \\
&= (a, b)
\end{aligned}
$$

- Note: EEs use $j = \sqrt{-1}$, the rest of the known universe uses $i = \sqrt{-1}$. Why is that?

How powerful are complex numbers?

- We now have a number system rich enough to solve arbitrary constant coefficient polynomial equations:

  $a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n = 0$

- If a's complex-valued, $n \geq 1$, and $a_0 \neq 0$, there are precisely n roots in complex number system

  "Fundamental Theorem of Algebra"

How do we represent these numbers?

- A positional number system lets us represent integers. E.g., in base 10:

  $xyz_{10} = x \cdot 10^2 + y \cdot 10^1 + z \cdot 10^0$

  $= x \cdot 100 + y \cdot 10 + z$

  x, y, z can each have 10 possible values: 0 to 9

Base 2 (binary) works the same way

$xyz_2 = x \cdot 2^2 + y \cdot 2^1 + z \cdot 2^0$

$= x \cdot 4 + y \cdot 2 + z$

x, y, z can each have 2 possible values: 0 or 1

each digit is called a "bit"

e.g.,

e.g.,	000	0
	001	1
	010	2
	011	3
	100	4
	101	5
	110	6
	111	7

Negative numbers

- With a fixed number of bits, one can represent negative numbers in a variety of ways.

  E.g., 4-bit binary number system:

- **unsigned** range 0 to 15 (0000 to 1111)

  unsigned integers with n bits range 0 to $2^n - 1$

- **offset** or **bias** (e.g., -7) range -7 to 8

  subtract fixed amount (such as midpoint value)

  generally bad for computation

4-bit Sign-Magnitude

1st bit encodes sign (0 = positive, 1 = negative)

bits 2, 3, 4 magnitude $\Rightarrow$ range 0 to 7 (000 to 111)

overall range -7 to +7

what about 1000? -0!

with n bits, use n-1 bits for magnitude

range $-(2^{n-1} - 1)$ to $+(2^{n-1} - 1)$

issues:

- two representations for "0", +0 and -0
- need significant hardware to support add, subtract

2's (radix) complement

- Use negative weight for 1st bit:

  $wxyz = w \cdot -(2)^3 + x \cdot 2^2 + y \cdot 2^1 + z \cdot 2^0$

  $= w \cdot -(8) + x \cdot 4 + y \cdot 2 + z$

- overall range -8 to +7
- 1st bit is still sign bit,

  with 0 = positive and 1 = negative
- only one zero: 0000

Properties of 2's complement

- least significant n-1 bits have unaltered meaning (i.e., standard positional notation and weights apply)
- most significant bit has weight negated (instead of weight 2^{n-1}, it is weight -2^{n-1})
- range $-(2^{n-1})$ to $+(2^{n-1}-1)$
- negation operation: flip all bits, add 1, throw away carry

-2 in 4-bit 2's complement notation?

binary	decimal
000	0
001	1
010	2
011	3
100	4
101	5
110	6
111	7