CSE 316A: Homework 5

Due on December 2, 2015
Total: 160 points

Notes

• There are 8 problems on 5 pages below, worth 20 points each (amounting to a total of 160 points). However, this homework will be graded out of 100 points, and any additional points you gain above the 100 points will count towards your final exam score (30% of your total score for the course), and none of it will go into your overall homework score.

• You have until 11:59 pm on Wednesday, December 2, to submit your homework, preferably electronically (by email or a private Piazza post).

• You must write / type all your answers on your own.

• Before you prepare your submission, you are free to discuss any problem with other students in this course and / or use resources other than the textbook and lecture notes. But you must clearly mention the students with whom you discussed the homework as well as any outside resources you used on your submission.

• Please review the academic integrity policy and grade appeal policy specified in the official course syllabus at http://www.cse.wustl.edu/~mithunchakraborty/SyllabusCSE316A.pdf.

• My office hours are Tuesdays 2:00 pm to 4:00 pm in Bryan 503.

Problems

1. (Textbook Chapter 2 Exercise 1, slightly modified) In any graph, we say that a node $X$ is pivotal for a pair of distinct nodes $Y$ and $Z$ if $X$ lies on every shortest path between $Y$ and $Z$ (and $X$ is neither $Y$ nor $Z$).

For example, in the graph in Figure 1, node $B$ is pivotal for two pairs: $(A, C)$ and $(A, D)$. Notice that $B$ is not pivotal for the pair $(D, E)$ since there are two different shortest paths connecting $D$ and $E$, one of which (using $C$ and $F$) doesn’t pass through $B$, so that $B$ is not on every shortest path between $D$ and $E$. On the other hand, node $D$ is not pivotal for any pairs.

(a) Give an example of a graph in which every node is pivotal for at least one pair of nodes. No explanation is necessary. (10 points)
(b) Give an example of a graph in which every node is pivotal for at least two different pairs of nodes. No explanation is necessary. (4 points)

(c) Give an example of a graph having at least four nodes in which there is a single node $X$ that is pivotal for every pair of nodes (not counting pairs that include $X$). No explanation is necessary. (6 points)

2. (Textbook Chapter 6 Exercise 10, slightly modified) In the bi-matrix representation of the game below, the rows correspond to player A’s strategies and the columns correspond to player B’s strategies. The first entry in each box is player A’s payoff and the second entry is player B’s payoff.

<table>
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<tr>
<th></th>
<th>B \</th>
<th>L</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>U</td>
<td>3, 3</td>
<td>1, 2</td>
<td></td>
</tr>
<tr>
<td>D</td>
<td>2, 1</td>
<td>3, 0</td>
<td></td>
</tr>
</tbody>
</table>

(a) Find all pure strategy Nash equilibria of this game. No explanation is necessary. (4 points)

(b) Notice from the payoff matrices above that Player A’s payoff from the pair of strategies $(U, L)$ is 3. Can you change player A’s payoff from this pair of strategies to some non-negative real number, keeping all other entries in the payoff matrices unchanged, in such a way that the resulting game has no pure strategy Nash equilibrium, even in a weak sense? Give a brief (1-3 sentence) explanation for your answer. (8 points)

(c) Now let’s go back to the original payoff matrices from part (a), where players A and B each get a payoff of 3 from the pair of strategies $(U, L)$, and ask an analogous question about player B. Can you change player B’s payoff from the pair of strategies $(U, L)$ to some non-negative real number, keeping all other entries in the payoff matrices unchanged, in such a way that the resulting game has no pure strategy Nash equilibrium, even in a weak sense? Give a brief (1-3 sentence) explanation for your answer. (8 points)

3. Consider a second-price sealed-bid auction where the seller has a single copy of an item that she values at 0, and everyone knows this.
There are three potential buyers for this copy of the item, Alice, Bob, and Charlie, whose personal valuations for the item are 10, 5, and 2 respectively. Everyone is selfish-rational, and that is common knowledge.

**Bidders are allowed to place bids that are positive integers only**, and of course everyone knows this information, too.

First, make sure that you know how the potential buyers will bid, and what the outcome of this auction format will be when the bidders do not know each others’ valuations at all (of course, each knows his / her own valuation). Now, tell me how the outcome will change, if at all, (specifically: what bid each bidder will place, who the winner will be, how much revenue the seller will earn, and how much surplus each of the three bidders will sustain) in each of the following scenarios:

(a) Alice knows Bob and Charlie’s valuations, and that any tie she is involved in will be broken in her favor; neither Bob nor Charlie knows anything about other bidders’ valuations. (**10 points**)

(b) Bob knows Alice and Charlie’s valuations, and that any tie he is involved in will be broken in his favor; neither Alice nor Charlie knows anything about other bidders’ valuations. (**10 points**)

4. Put yourself in the shoes of an average wizard / witch in the period between June, 1995, and June, 1996. Based on the repeated reassurances of the Ministry of Magic, you have always believed that there is a very small probability that the evil wizard Lord Voldemort, who has been mostly powerless since 1981, will return to full power. Let this probability be \( \varepsilon \), \( 0 < \varepsilon \ll 0.5 \). However, the highly respected wizard Albus Dumbledore maintains that Voldemort is back and must be dealt with. You think that in any situation, there is a probability \((1 - \varepsilon)\) that Dumbledore is speaking the truth.\(^1\) To what value should you update your personal probability that Voldemort is back? How does your revised probability depend on \( \varepsilon \)?

5. Suppose we are studying the diffusion of an innovation \( A \) over a social network where the status quo is denoted by \( B \). Consider the following bi-matrix representation of the networked coordination game being played on every edge, i.e. between any pair of neighbors \((u, v)\) in this network\(^2\):

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>( a, a )</td>
<td>( \delta, \varepsilon )</td>
</tr>
<tr>
<td>B</td>
<td>( \varepsilon, \delta )</td>
<td>( b, b )</td>
</tr>
</tbody>
</table>

\(^1\)Interpret this to mean that the probability that Dumbledore says that Voldemort is back given that the latter is indeed back is \((1 - \varepsilon)\), and so is the probability that Dumbledore says that Voldemort is not back given that he has actually not returned.

\(^2\)Ref: Textbook Chapter 19 pages 566-568
Here \( a, b, \delta, \varepsilon > 0 \), and \( \max(\delta, \varepsilon) < \min(a, b) \). Is it still possible to describe the diffusion in terms of a threshold-based behavior switching rule, as discussed in class? If yes, derive the diffusion threshold for this network. If no, give a short explanation.

6. Consider the diffusion of an innovation \( A \) over the network depicted in Figure 2 where the status quo is denoted by \( B \), and the shaded nodes \( v \) and \( w \) are the initial adopters.

![Figure 2: Network for Problem 6.](image)

Suppose that a traditional networked coordination game is being played on every edge in this network (Textbook Chapter 19 Figure 19.1, or equivalently the game matrix in Problem 5 with \( a, b > 0 \) and \( \delta = 0, \varepsilon = 0 \)) but you do not know what the values of \( a \) and \( b \) are. Let \( r \) denote \( a/b \), the payoff ratio of \( A \) to \( B \).

However, you observe that as the cascade proceeds, nodes \( r \) and \( t \) switch over (from \( B \) to \( A \)) at the end of the first round, and then the cascade stops (with \( s \) and \( u \) still engaging in \( B \)). From this information alone, derive the range of possible values of \( r \).

7. (Textbook Chapter 19 Exercise 4, slightly modified) Consider the threshold-based model for the diffusion of a new behavior through a social network, described in class. Suppose that initially everyone is using behavior \( B \) in the social network in Figure 3, and then a new behavior \( A \) is introduced. This behavior has a threshold of \( q = 1/2 \): any node will switch to \( A \) if at least \( 1/2 \) of its neighbors are using it.

(a) Find a set of three nodes in the network with the property that if they act as the three initial adopters of \( A \), then it will spread to all nodes. (In other words, three nodes who are capable of causing a complete cascade of adoptions of \( A \) for a threshold of \( 1/2 \) on this network.) (5 points)

(b) Is the set of three nodes you found in (a) the only set of three initial adopters capable of causing a complete cascade of \( A \), or can you find a different set of three initial adopters who could also cause a complete cascade of \( A \)? (5 points)

(c) Find three clusters in the network, each of density greater than \( 1/2 \), with the property that no node belongs to more than one of these clusters. (5 points)

(d) How does your answer to (c) help explain why there is no set of initial adopters consisting of only two nodes in the network that would be capable of causing a complete cascade of adoptions of \( A \)? (5 points)

8. Consider the collective action problem in the presence of pluralistic ignorance (Textbook Chapter 19 Section 19.6) for the social network in Figure 4. The integer next to each node
represents her personal participation threshold; e.g. \( u \) will join in with the action only if she is convinced in her mind that at least 3 people (including herself) will participate. Every node knows the entire network structure but the thresholds of her neighbors only, and everyone uses the same line of reasoning — all of the above is common knowledge. Do you think the event can occur (i.e. will at least one person be comfortable taking part in it)? Give a very brief explanation.