Computer Systems
Performance Analysis:
Design of Experiments

Raj Jain
Washington University in Saint Louis
Saint Louis, MO 63130, USA
Jain@cse.wustl.edu

A Tutorial given at XXIX Brazilian Symposium on Computer Networks and Distributed Systems,
May 30-June 3, 2011, Campo Grande, Brazil

The Audio/Video recordings of this tutorial are available at:

http://www.cse.wustl.edu/~jain/tutorials/ied_tut.htm
Performance Analysis

- Performance = Measurement, Simulation, Analytical Modeling
- Both measurement and simulation require resources and time
- Performance is affected by many factors:
  - For example: Network appliance performance is affected by CPU, Disk, network card, packet sizes
- Each of these factors can have several levels: For example:
  - 3 types of CPUs: Single core, dual core, multicore
  - 4 types of disks: 4800 rpm, 5200 rpm, 7200 rpm, 10000 rpm
  - 2 types of network: 10 Mbps, 100 Mbps, 1 Gbps, 10 Gbps
  - 6 packet sizes: 64B, 128KB, 512B, 1024B, 1518B, 9KB
- How many experiments do we need? $3 \times 4 \times 2 \times 6 = 144$
- What is the effect of CPU?
Experimental Design

- Design a proper set of experiments for measurement or simulation. Don’t need to do all possible combinations.
- Develop a model that best describes the data obtained.
- Estimate the contribution of each factor to the performance.
- Isolate the measurement errors.
- Estimate confidence intervals for model parameters.
- Check if the alternatives are significantly different.
- Check if the model is adequate.
- The techniques apply to all systems: Networks, Distributed Systems, Data bases, algorithms, …
Text Book

Overview

1. Introduction to Design of Experiments
2. $2^k$ Factorial Designs
3. $2^{kr}$ Factorial Designs
4. $2^{k-p}$ Fractional Factorial Designs
Module 1: Introduction to Design of Experiments
Overview

- What is experimental design?
- Terminology
- Common mistakes
- Sample designs
Terminology

- **Factors**: Variables that affect the response variable. E.g., CPU type, memory size, number of disk drives, workload used, and user's educational level. Also called predictor variables or predictors.

- **Levels**: The values that a factor can assume, E.g., the CPU type has three levels: 68000, 8080, or Z80. # of disk drives has four levels. Also called *treatment*.

- **Replication**: Repetition of all or some experiments.

- **Design**: The number of experiments, the factor level and number of replications for each experiment. E.g., Full Factorial Design with 5 replications: $3 \times 3 \times 4 \times 3 \times 3$ or 324 experiments, each repeated five times.
Terminology (Cont)

- Interaction \Rightarrow Effect of one factor depends upon the level of the other.

Table 1: Noninteracting Factors

<table>
<thead>
<tr>
<th></th>
<th>$A_1$</th>
<th>$A_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B_1$</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>$B_2$</td>
<td>6</td>
<td>8</td>
</tr>
</tbody>
</table>

Table 2: Interacting Factors

<table>
<thead>
<tr>
<th></th>
<th>$A_1$</th>
<th>$A_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B_1$</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>$B_2$</td>
<td>6</td>
<td>9</td>
</tr>
</tbody>
</table>
Common Mistakes in Experimentation

- The variation due to experimental error is ignored.
- Important parameters are not controlled.
- Effects of different factors are not isolated.
- Simple one-factor-at-a-time designs are used.
- Interactions are ignored.
- Too many experiments are conducted.

Better: two phases.
Types of Experimental Designs

- **Simple Designs**: Vary one factor at a time
  
  \[ \text{# of Experiments} = 1 + \sum_{i=1}^{k} (n_i - 1) \]
  
  - Not statistically efficient.
  - Wrong conclusions if the factors have interaction.
  - Not recommended.

- **Full Factorial Design**: All combinations.
  
  \[ \text{# of Experiments} = \prod_{i=1}^{k} n_i \]
  
  - Can find the effect of all factors.
  - Too much time and money.
  - May try $2^k$ design first.
Types of Experimental Designs (Cont)

- Fractional Factorial Designs: Less than Full Factorial
  - Save time and expense.
  - Less information.
  - May not get all interactions.
  - Not a problem if negligible interactions
Example

Personal workstation design
1. Processor: 68000, Z80, or 8086.
2. Memory size: 512K, 2M, or 8M bytes
3. Number of Disks: One, two, three, or four
4. Workload: Secretarial, managerial, or scientific.
5. User education: High school, college, or post-graduate level.

Five **Factors** at 3x3x4x3x3 levels
A Sample Fractional Factorial Design

- **Workstation Design:**
  - (3 CPUs)(3 Memory levels)(3 workloads)(3 ed levels)
  - = 81 experiments

<table>
<thead>
<tr>
<th>Experiment Number</th>
<th>CPU</th>
<th>Memory Level</th>
<th>Workload Type</th>
<th>Educational Level</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>68000</td>
<td>512K</td>
<td>Managerial</td>
<td>High School</td>
</tr>
<tr>
<td>2</td>
<td>68000</td>
<td>2M</td>
<td>Scientific</td>
<td>Post-graduate</td>
</tr>
<tr>
<td>3</td>
<td>68000</td>
<td>8M</td>
<td>Secretarial</td>
<td>College</td>
</tr>
<tr>
<td>4</td>
<td>Z80</td>
<td>512K</td>
<td>Scientific</td>
<td>College</td>
</tr>
<tr>
<td>5</td>
<td>Z80</td>
<td>2M</td>
<td>Secretarial</td>
<td>High School</td>
</tr>
<tr>
<td>6</td>
<td>Z80</td>
<td>8M</td>
<td>Managerial</td>
<td>Post-graduate</td>
</tr>
<tr>
<td>7</td>
<td>8086</td>
<td>512K</td>
<td>Secretarial</td>
<td>Post-graduate</td>
</tr>
<tr>
<td>8</td>
<td>8086</td>
<td>2M</td>
<td>Managerial</td>
<td>College</td>
</tr>
<tr>
<td>9</td>
<td>8086</td>
<td>8M</td>
<td>Scientific</td>
<td>High School</td>
</tr>
</tbody>
</table>
Goal of proper experimental design is to get the maximum information with minimum number of experiments

Factors, levels, full-factorial designs
Module 2: 
$2^k$ Factorial Designs
Overview

- $2^2$ Factorial Designs
- Model
- Computation of Effects
- Sign Table Method
- Allocation of Variation
- General $2^k$ Factorial Designs
2\(^k\) Factorial Designs

- \(k\) factors, each at two levels.
- Easy to analyze.
- Helps in sorting out impact of factors.
- Good at the beginning of a study.
- Valid only if the effect is unidirectional.
  E.g., memory size, the number of disk drives
2^2 Factorial Designs

- Two factors, each at two levels.

<table>
<thead>
<tr>
<th>Cache Size</th>
<th>Memory Size</th>
<th>Performance in MIPS</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>4M Bytes</td>
<td>16M Bytes</td>
</tr>
<tr>
<td>1K</td>
<td>15</td>
<td>45</td>
</tr>
<tr>
<td>2K</td>
<td>25</td>
<td>75</td>
</tr>
</tbody>
</table>

\[ x_A = \begin{cases} -1 & \text{if 4M bytes memory} \\ 1 & \text{if 16M bytes memory} \end{cases} \]

\[ x_B = \begin{cases} -1 & \text{if 1K bytes cache} \\ 1 & \text{if 2K bytes cache} \end{cases} \]
Model

\[ y = q_0 + q_A x_A + q_B x_B + q_{AB} x_A x_B \]

Observations:

\[ 15 = q_0 - q_A - q_B + q_{AB} \]
\[ 45 = q_0 + q_A - q_B - q_{AB} \]
\[ 25 = q_0 - q_A + q_B - q_{AB} \]
\[ 75 = q_0 + q_A + q_B + q_{AB} \]

Solution:

\[ y = 40 + 20x_A + 10x_B + 5x_A x_B \]

**Interpretation:** Mean performance = 40 MIPS
Effect of memory = 20 MIPS; Effect of cache = 10 MIPS
Interaction between memory and cache = 5 MIPS.
## Sign Table Method

<table>
<thead>
<tr>
<th>I</th>
<th>A</th>
<th>B</th>
<th>AB</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>15</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>45</td>
</tr>
<tr>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>25</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>75</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>160</th>
<th>80</th>
<th>40</th>
<th>20</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>40</td>
<td>20</td>
<td>10</td>
<td>5</td>
<td>Total/4</td>
</tr>
</tbody>
</table>
Allocation of Variation

- Importance of a factor = proportion of the \textit{variation} explained

\[ \text{Sample Variance of } y = s_y^2 = \frac{\sum_{i=1}^{2^2} (y_i - \bar{y})^2}{2^2 - 1} \]

\[ \text{Total Variation of } y = \text{SST} = \sum_{i=1}^{2^2} (y_i - \bar{y})^2 \]

- For a $2^2$ design:

\[ \text{SST} = 2^2 q_A^2 + 2^2 q_B^2 + 2^2 q_{AB}^2 = \text{SSA} + \text{SSB} + \text{SSAB} \]

- Variation due to A = SSA = $2^2 q_A^2$
- Variation due to B = SSB = $2^2 q_B^2$
- Variation due to interaction = SSAB = $2^2 q_{AB}^2$
- Fraction explained by A = $\frac{\text{SSA}}{\text{SST}}$  \hspace{1cm} \text{Variation} \neq \text{Variance}
Example 17.2

- Memory-cache study:

\[ \bar{y} = \frac{1}{4} (15 + 55 + 25 + 75) = 40 \]

Total Variation = \( \sum_{i=1}^{4} (y_i - \bar{y})^2 \)

= \((25^2 + 15^2 + 15^2 + 35^2)\)

= 2100

= \(4 \times 20^2 + 4 \times 10^2 + 4 \times 5^2\)

- Total variation = 2100
  - Variation due to Memory = 1600 (76%)
  - Variation due to cache = 400 (19%)
  - Variation due to interaction = 100 (5%)
Case Study 17.1: Interconnection Nets

- Memory interconnection networks: Omega and Crossbar.
- Memory reference patterns: Random and Matrix
- Fixed factors:
  - Number of processors was fixed at 16.
  - Queued requests were not buffered but blocked.
  - Circuit switching instead of packet switching.
  - Random arbitration instead of round robin.
  - Infinite interleaving of memory ⇒ no memory bank contention.
# 2² Design for Interconnection Networks

Factors Used in the Interconnection Network Study

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Factor</th>
<th>Level</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>Type of the network</td>
<td>Crossbar</td>
</tr>
<tr>
<td>B</td>
<td>Address Pattern Used</td>
<td>Omega</td>
</tr>
<tr>
<td></td>
<td>Random</td>
<td>Mother</td>
</tr>
<tr>
<td></td>
<td>Matrix</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th></th>
<th>90% Transit N</th>
<th>Response R</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>B</td>
<td>Throughput T</td>
<td></td>
</tr>
<tr>
<td>-1</td>
<td>-1</td>
<td>0.0641</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>-1</td>
<td>0.4220</td>
<td>5</td>
</tr>
<tr>
<td>-1</td>
<td>1</td>
<td>0.7922</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0.4717</td>
<td>4</td>
</tr>
</tbody>
</table>
Interconnection Networks Results

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Mean Estimate</th>
<th>Variation Explained</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>T</td>
<td>N</td>
</tr>
<tr>
<td>( q_0 )</td>
<td>0.5725</td>
<td>3.5</td>
</tr>
<tr>
<td>( q_A )</td>
<td>0.0595</td>
<td>-0.5</td>
</tr>
<tr>
<td>( q_B )</td>
<td>-0.1257</td>
<td>1.0</td>
</tr>
<tr>
<td>( q_{AB} )</td>
<td>-0.0346</td>
<td>0.0</td>
</tr>
</tbody>
</table>

- Average throughput = 0.5725
- Most effective factor = B = Reference pattern
  ⇒ The address patterns chosen are very different.
- Reference pattern explains \( \mp 0.1257 \) (77%) of variation.
- Effect of network type = 0.0595
  Omega networks = Average + 0.0595
  Crossbar networks = Average - 0.0595
- Slight interaction (0.0346) between reference pattern and network type.
General $2^k$ Factorial Designs

- $k$ factors at two levels each.
  - $2^k$ experiments.
  - $2^k$ effects:
    - $k$ main effects
    - $\binom{k}{2}$ two factor interactions
    - $\binom{k}{3}$ three factor interactions...
2^k Design Example

- Three factors in designing a machine:
  - Cache size
  - Memory size
  - Number of processors

<table>
<thead>
<tr>
<th>Factor</th>
<th>Level -1</th>
<th>Level 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>A Memory Size</td>
<td>4MB</td>
<td>16MB</td>
</tr>
<tr>
<td>B Cache Size</td>
<td>1kB</td>
<td>2kB</td>
</tr>
<tr>
<td>C Number of Processors</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>

\[ y = q_0 + q_A x_A + q_B x_B + q_C x_C + q_{AB} x_A x_B + q_{AC} x_A x_C + q_{BC} x_B x_C + q_{ABC} x_A x_B x_C \]
### Design Example (cont)

<table>
<thead>
<tr>
<th>Cache Size</th>
<th>4M Bytes</th>
<th>16M Bytes</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1 Proc</td>
<td>2 Proc</td>
</tr>
<tr>
<td>1K Byte</td>
<td>14</td>
<td>46</td>
</tr>
<tr>
<td>2K Byte</td>
<td>10</td>
<td>50</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>I</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>AB</th>
<th>AC</th>
<th>BC</th>
<th>ABC</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>14</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>22</td>
</tr>
<tr>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>10</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>34</td>
</tr>
<tr>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>46</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>58</td>
</tr>
<tr>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>50</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>86</td>
</tr>
<tr>
<td>320</td>
<td>80</td>
<td>40</td>
<td>160</td>
<td>40</td>
<td>16</td>
<td>24</td>
<td>9</td>
<td>Total</td>
</tr>
<tr>
<td>40</td>
<td>10</td>
<td>5</td>
<td>20</td>
<td>5</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td>Total/8</td>
</tr>
</tbody>
</table>


Analysis of $2^k$ Design

$$
\text{SST} = 2^3 (q_A^2 + q_B^2 + q_C^2 + q_{AB}^2 + q_{AC}^2 + q_{BC}^2 + q_{ABC}^2) \\
= 8(10^2 + 5^2 + 20^2 + 5^2 + 2^2 + 3^2 + 1^2) \\
= 800 + 200 + 3200 + 200 + 32 + 72 + 8 = 4512 \\
= 18\% + 4\% + 71\% + 4\% + 1\% + 2\% + 0\% \\
= 100\% \\
$$

- Number of Processors ($C$) is the most important factor.
Summary

- $2^k$ design allows $k$ factors to be studied at two levels each
- Can compute main effects and all multi-factors interactions
- Easy computation using sign table method
- Easy allocation of variation using squares of effects
Module 3: $2^k r$ Factorial Designs
Overview

- Computation of Effects
- Estimation of Experimental Errors
- Allocation of Variation
- Confidence Intervals for Effects
- Confidence Intervals for Predicted Responses
- Visual Tests for Verifying the assumptions
- Multiplicative Models
2<sup>kr</sup> Factorial Designs

- *r* replications of 2<sup>k</sup> Experiments
  ⇒ 2<sup>kr</sup> observations.
  ⇒ Allows estimation of experimental errors.

- Model:
  \[ y = q_0 + q_A x_A + q_B x_B + q_{AB} x_A x_B + e \]

- e = Experimental error
Computation of Effects

Simply use means of r measurements

<table>
<thead>
<tr>
<th>I</th>
<th>A</th>
<th>B</th>
<th>A</th>
<th>B</th>
<th>y</th>
<th>Mean</th>
<th>( \bar{y} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td></td>
<td>(15, 18, 12)</td>
<td>15</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td></td>
<td>(45, 48, 51)</td>
<td>48</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td></td>
<td>(25, 28, 19)</td>
<td>24</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
<td>(75, 75, 81)</td>
<td>77</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>164</th>
<th>86</th>
<th>38</th>
<th>20</th>
<th>total</th>
</tr>
</thead>
<tbody>
<tr>
<td>41</td>
<td>21.5</td>
<td>9.5</td>
<td>5</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Effects: \( q_0 = 41 \), \( q_A = 21.5 \), \( q_B = 9.5 \), \( q_{AB} = 5 \).
Experimental Errors: Example

- Estimated Response:
  \[ \hat{y}_1 = q_0 - q_A - q_B + q_{AB} = 41 - 21.5 - 9.5 + 5 = 15 \]

- Experimental errors:
  \[ e_{11} = y_{11} - \hat{y}_1 = 15 - 15 = 0 \]

<table>
<thead>
<tr>
<th>i</th>
<th>Effect</th>
<th>Estimated Response</th>
<th>Measured Responses</th>
<th>Errors</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>I</td>
<td>A</td>
<td>B</td>
<td>A</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>
 Allocation of Variation

- Total variation or total sum of squares:

\[ \text{SST} = \sum_{i,j} (y_{ij} - \bar{y}..)^2 \]

\[ y_{ij} = q_0 + q_A x_{Ai} + q_B x_{Bi} + q_{AB} x_{Ai} x_{Bi} + e_{ij} \]

\[ \sum_{i,j} (y_{ij} - \bar{y}..)^2 = 2^2 r q_A^2 + 2^2 r q_B^2 + 2^2 r q_{AB}^2 + \sum_{i,j} e_{ij}^2 \]

<table>
<thead>
<tr>
<th>Term</th>
<th>Value</th>
<th>Percentage</th>
</tr>
</thead>
<tbody>
<tr>
<td>SST</td>
<td>7032</td>
<td>78.88%</td>
</tr>
<tr>
<td>SSA</td>
<td>5547</td>
<td>15.4%</td>
</tr>
<tr>
<td>SSB</td>
<td>1083</td>
<td>4.27%</td>
</tr>
<tr>
<td>SSAB</td>
<td>300</td>
<td>1.45%</td>
</tr>
<tr>
<td>SSE</td>
<td>102</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>100%</td>
<td></td>
</tr>
</tbody>
</table>
Confidence Intervals For Effects

- Effects are random variables.
- Errors $\sim N(0, \sigma_e) \Rightarrow y \sim N(\bar{y}, \sigma_e)$
- Variance of errors:
  $$s_e^2 = \frac{1}{2^2(r-1)} \sum_{i,j} e_{ij}^2 = \frac{\text{SSE}}{2^2(r-1)} \triangle \text{MSE}$$
- Similarly,
  $$S_{qA} = S_{qB} = S_{qAB} = \frac{s_e}{\sqrt{2^2 r}}$$
- Confidence intervals (CI) for the effects:
  $$q_i \mp t[1-\alpha/2;2^2(r-1)] S_{q_i}$$
- CI does not include a zero $\Rightarrow$ significant
Example 18.4

- For Memory-cache study: Standard deviation of errors:
  \[ s_e = \sqrt{\frac{SSE}{2^2(r - 1)}} = \sqrt{\frac{102}{8}} = \sqrt{12.75} = 3.57 \]

- Standard deviation of effects:
  \[ s_{q_i} = \frac{s_e}{\sqrt{(2^2r)}} = \frac{3.57}{\sqrt{12}} = 1.03 \]

- For 90% Confidence: \( t_{[0.95,8]} = 1.86 \)

- Confidence intervals: \( q_i \mp (1.86)(1.03) = q_i \mp 1.92 \)
  \[ q_0 = (39.08, 42.91) \]
  \[ q_A = (19.58, 23.41) \]
  \[ q_B = (7.58, 11.41) \]
  \[ q_{AB} = (3.08, 6.91) \]

- No zero crossing \( \Rightarrow \) All effects are significant.
Assumptions

1. Errors are statistically independent.
2. Errors are additive.
3. Errors are normally distributed.
4. Errors have a constant standard deviation $\sigma_e$.
5. Effects of factors are additive
   $\Rightarrow$ observations are independent and normally distributed with constant variance.
Visual Tests

1. Independent Errors:
   - Scatter plot of residuals versus the predicted response $\hat{Y}_i$
   - Magnitude of residuals $<$ Magnitude of responses/10 $\Rightarrow$ Ignore trends
   - Plot the residuals as a function of the experiment number
   - Trend up or down $\Rightarrow$ other factors or side effects

2. Normally distributed errors:
   Normal quantile-quantile plot of errors

3. Constant Standard Deviation of Errors:
   Scatter plot of y for various levels of the factor
   Spread at one level significantly different than that at other
   $\Rightarrow$ Need transformation
Example 18.7: Memory-cache
Multiplicative Models

- Additive model:
  \[ y_{ij} = q_0 + q_A x_A + q_B x_B + q_{AB} x_A x_B + e_{ij} \]

- Not valid if effects do not add.
  E.g., execution time of workloads.
  \( i \)th processor speed = \( v_i \) instructions/second.
  \( j \)th workload Size = \( w_j \) instructions

- The two effects multiply. Logarithm \( \Rightarrow \) additive model:
  Execution Time \( y_{ij} = v_i \times w_j \)
  \[ \log(y_{ij}) = \log(v_i) + \log(w_j) \]

- Correct Model:
  \[ y'_{ij} = q_0 + q_A x_A + q_B x_B + q_{AB} x_A x_B + e_{ij} \]
  Where, \( y'_{ij} = \log(y_{ij}) \)
Multiplicative Model (Cont)

- Taking an antilog of effects:
  \[ u_A = 10^{q_A}, \quad u_B = 10^{q_B}, \quad \text{and} \quad u_{AB} = 10^{q_{AB}} \]

- \( u_A \) = ratio of MIPS rating of the two processors
- \( u_B \) = ratio of the size of the two workloads.
- Antilog of additive mean \( q_0 \) \( \Rightarrow \) geometric mean
  \[ \hat{y} = 10^{q_0} = \left( y_1 y_2 \cdots y_n \right)^{1/n} \quad n = 2^r \]
Example 18.8: Execution Times

Analysis Using an Additive Model

<table>
<thead>
<tr>
<th>I</th>
<th>A</th>
<th>B</th>
<th>AB</th>
<th>(y)</th>
<th>Mean (\bar{y})</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>(85.10, 79.50, 147.90)</td>
<td>104.170</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>(0.891, 1.047, 1.072)</td>
<td>1.003</td>
</tr>
<tr>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>(0.955, 0.933, 1.122)</td>
<td>1.003</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>(0.0148, 0.0126, 0.0118)</td>
<td>0.013</td>
</tr>
<tr>
<td></td>
<td>106.19</td>
<td>-104.15</td>
<td>-104.15</td>
<td>102.17</td>
<td>total</td>
</tr>
<tr>
<td></td>
<td>26.55</td>
<td>-26.04</td>
<td>-26.04</td>
<td>25.54</td>
<td>total/4</td>
</tr>
</tbody>
</table>

Additive model is not valid because:

- Physical consideration \(\Rightarrow\) effects of workload and processors do not add. They multiply.
- Large range for \(y\). \(y_{\text{max}}/y_{\text{min}} = 147.90/0.0118\) or 12,534
  \(\Rightarrow\) log transformation
- Taking an arithmetic mean of 114.17 and 0.013 is inappropriate.
Example 18.8 (Cont)

- The residuals are not small as compared to the response.

- The spread of residuals is large at larger value of the response.  
  \[ \Rightarrow \text{log transformation} \]
Example 18.8 (Cont)

- Residual distribution has a longer tail than normal
## Analysis Using Multiplicative Model

Data After Log Transformation

<table>
<thead>
<tr>
<th>I</th>
<th>A</th>
<th>B</th>
<th>AB</th>
<th>y</th>
<th>Mean</th>
<th>( \bar{y} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>( 1.93, 1.90, 2.17)</td>
<td>2.00</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>( -0.05, 0.02, 0.03)</td>
<td>0.00</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>( -0.02, -0.03, 0.05)</td>
<td>0.00</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>( -1.83, -1.90, -1.93)</td>
<td>-1.89</td>
<td></td>
</tr>
<tr>
<td>0.11</td>
<td>-3.89</td>
<td>-3.89</td>
<td>0.11</td>
<td>total</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.03</td>
<td>-0.97</td>
<td>-0.97</td>
<td>0.03</td>
<td>total/4</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
## Variation Explained by the Two Models

<table>
<thead>
<tr>
<th>Factor</th>
<th>Additive Model</th>
<th>Multiplicative Model</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Effect</td>
<td>% Var.</td>
</tr>
<tr>
<td>I</td>
<td>26.55</td>
<td>(16.35, 36.74)</td>
</tr>
<tr>
<td>A</td>
<td>-26.04</td>
<td>30.1%</td>
</tr>
<tr>
<td>B</td>
<td>-26.04</td>
<td>30.1%</td>
</tr>
<tr>
<td>AB</td>
<td>25.54</td>
<td>29.0%</td>
</tr>
<tr>
<td>e</td>
<td>10.8%</td>
<td></td>
</tr>
</tbody>
</table>

† ⇒ Not Significant

- **With multiplicative model:**
  - Interaction is almost zero.
  - Unexplained variation is only 0.2%
Visual Tests

- Conclusion: Multiplicative model is better than the additive model.
Interpretation of Results

\[
\log(y) = q_0 + q_A x_A + q_B x_B + q_{AB} x_A x_B + e
\]

\[\Rightarrow y = 10^{q_0} 10^{q_A x_A} 10^{q_B x_B} 10^{q_{AB} x_A x_B} 10^e\]

\[= 10^{0.03} 10^{-0.97 x_A} 10^{-0.97 x_B} 10^{0.03 x_A x_B} 10^e\]

\[= 1.07 \times 0.107^{x_A} \times 0.107^{x_B} \times 1.07^{x_A x_B} 10^e\]

- The time for an average processor on an average benchmark is 1.07.
- The time on processor A1 is nine times (0.107⁻¹) that on an average processor. The time on A2 is one ninth (0.107¹) of that on an average processor.
- MIPS rate for A2 is 81 times that of A1.
- Benchmark B1 executes 81 times more instructions than B2.
- The interaction is negligible.

\[\Rightarrow \text{Results apply to all benchmarks and processors.}\]
Replications allow estimation of measurement errors ⇒ Confidence Intervals of parameters
  Allocation of variation is proportional to square of effects
Multiplicative models are appropriate if the factors multiply
Visual tests for independence normal errors
Module 4: \(2^{k-p}\) Fractional Factorial Designs
Overview

- $2^{k-p}$ Fractional Factorial Designs
- Sign Table for a $2^{k-p}$ Design
- Confounding
- Other Fractional Factorial Designs
- Algebra of Confounding
- Design Resolution
2^{k-p} Fractional Factorial Designs

- Large number of factors
  - ⇒ large number of experiments
  - ⇒ full factorial design too expensive
  - ⇒ Use a fractional factorial design

- 2^{k-p} design allows analyzing k factors with only 2^{k-p} experiments.
  - 2^{k-1} design requires only half as many experiments
  - 2^{k-2} design requires only one quarter of the experiments
## Example: $2^{7-4}$ Design

<table>
<thead>
<tr>
<th>Expt No.</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
<th>G</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>5</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>7</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>8</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

- Study 7 factors with only 8 experiments!
Fractional Design Features

- Full factorial design is easy to analyze due to orthogonality of sign vectors.

Fractional factorial designs also use orthogonal vectors. That is:
  
  - The sum of each column is zero.
    \[ \sum_i x_{ij} = 0 \ \forall \ j \]
    
    \( j \)th variable, \( i \)th experiment.
  
  - The sum of the products of any two columns is zero.
    \[ \sum_i x_{ij}x_{il} = 0 \ \forall \ j \neq l \]
  
  - The sum of the squares of each column is \( 2^{7-4} \), that is, 8.
    \[ \sum_i x_{ij}^2 = 8 \ \forall \ j \]
## Analysis of Fractional Factorial Designs

<table>
<thead>
<tr>
<th></th>
<th>I</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
<th>G</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>20</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>35</td>
</tr>
<tr>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>7</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>42</td>
</tr>
<tr>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>36</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>50</td>
</tr>
<tr>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>45</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>82</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>317</th>
<th>101</th>
<th>35</th>
<th>109</th>
<th>43</th>
<th>1</th>
<th>47</th>
<th>3</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>39.62</td>
<td>12.62</td>
<td>4.37</td>
<td>13.62</td>
<td>5.37</td>
<td>0.125</td>
<td>5.87</td>
<td>0.37</td>
<td>Total/8</td>
</tr>
</tbody>
</table>

Factors A through G explain 37.26%, 4.74%, 43.40%, 6.75%, 0%, 8.06%, and 0.03% of variation, respectively.

⇒ Use only factors C and A for further experimentation.
Sign Table for a $2^{k-p}$ Design

Steps:
1. Prepare a sign table for a full factorial design with $k-p$ factors.
2. Mark the first column I.
3. Mark the next $k-p$ columns with the $k-p$ factors.
4. Of the $(2^{k-p}-k-p-1)$ columns on the right, choose $p$ columns and mark them with the $p$ factors which were not chosen in step 1.
## Example: $2^{7-4}$ Design

<table>
<thead>
<tr>
<th>Expt No.</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>AB</th>
<th>AC</th>
<th>BC</th>
<th>ABC</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>5</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>7</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>8</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>
## Example: $2^{4-1}$ Design

<table>
<thead>
<tr>
<th>Expt No.</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>AB</th>
<th>AC</th>
<th>BC</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>5</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>7</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>8</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>
Confounding

Confounding: Only the combined influence of two or more effects can be computed.

\[ q_A = \sum_i y_i x_{Ai} \]
\[ = \frac{-y_1 + y_2 - y_3 + y_4 - y_5 + y_6 - y_7 + y_8}{8} \]

\[ q_D = \sum_i y_i x_{Di} \]
\[ = \frac{-y_1 + y_2 + y_3 - y_4 + y_5 - y_6 - y_7 + y_8}{8} \]
Confounding (Cont)

\[ q_{ABC} = \sum_i y_i x_A x_B x_C i \]
\[ = \frac{-y_1 + y_2 + y_3 - y_4 + y_5 - y_6 - y_7 + y_8}{8} \]

\[ q_D = q_{ABC} \]

\[ q_D + q_{ABC} = \sum_i y_i x_A x_B x_C i \]
\[ = \frac{-y_1 + y_2 + y_3 - y_4 + y_5 - y_6 - y_7 + y_8}{8} \]

\[ \Rightarrow \text{Effects of D and ABC are confounded. Not a problem if } q_{ABC} \text{ is negligible.} \]
Confounding (Cont)

- Confounding representation: \( D = ABC \)

Other Confoundings:

\[
q_A = q_{BCD} = \sum_i y_i x_{Ai} \\
= \frac{-y_1 + y_2 - y_3 + y_4 - y_5 + y_6 - y_7 + y_8}{8}
\]

\( \Rightarrow A = BCD \)

\( A = BCD, \ B = ACD, \ C = ABD, \ AB = CD, \ AC = BD, \ BC = AD, \ ABC = D, \) and \( I = ABCD \)

- \( I = ABCD \Rightarrow \) confounding of ABCD with the mean.
Other Fractional Factorial Designs

- A fractional factorial design is not unique. $2^p$ different designs.

Another $2^{4-1}$ Experimental Design

<table>
<thead>
<tr>
<th>Expt No.</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>AC</th>
<th>BC</th>
<th>ABC</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>5</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>7</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>8</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

- Confoundings: $I=ABD$, $A=BD$, $B=AD$, $C=ABCD$, $D=AB$, $AC=BCD$, $BC=ACD$, $ABC=CD$

Not as good as the previous design.
Summary

- Fractional factorial designs allow a large number of variables to be analyzed with a small number of experiments.
- Many effects and interactions are confounded.
Other Designs

- One factor with many levels
e.g., 1 factor with 5 levels

- Two factors with different levels,
e.g., 2 factors with 4×5 levels

- Multiple factors with different levels,
e.g., 4 factors with 3×4×5×2 levels

- All these designs and others are discussed in the book.
Overall Summary

- $2^k$ design allows $k$ factors to be studied at two levels each
- Can compute main effects and all multi-factors interactions
- Easy computation using sign table method
- Easy allocation of variation using squares of effects
- $2^{kr}$ design with replications allow estimation of measurement errors ⇒ Confidence Intervals of parameters
- Multiplicative models are appropriate if the factors multiply
- Visual tests for independence normal errors
- $2^{k-p}$ Fractional factorial designs allow a large number of variables to be analyzed with a small number of experiments
- Many effects and interactions are confounded