$2^k r$ Factorial Designs

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These slides are available on-line at:
http://www.cse.wustl.edu/~jain/cse567-17/
Overview

- Computation of Effects
- Estimation of Experimental Errors
- Allocation of Variation
- Confidence Intervals for Effects
- Confidence Intervals for Predicted Responses
- Visual Tests for Verifying the assumptions
- Multiplicative Models
2^{kr} \text{ Factorial Designs}

- \( r \) replications of \( 2^k \) Experiments
  \[ \Rightarrow 2^{kr} \text{ observations.} \]
  \[ \Rightarrow \text{Allows estimation of experimental errors.} \]

- Model:
  \[ y = q_0 + q_A x_A + q_B x_B + q_{AB} x_A x_B + e \]

- \( e = \text{Experimental error} \)
## Computation of Effects

Simply use means of r measurements

<table>
<thead>
<tr>
<th>I</th>
<th>A</th>
<th>B</th>
<th>A</th>
<th>B</th>
<th>y</th>
<th>Mean</th>
<th>$\bar{y}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>(15, 18, 12)</td>
<td>15</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>(45, 48, 51)</td>
<td>48</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>(25, 28, 19)</td>
<td>24</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>(75, 75, 81)</td>
<td>77</td>
<td></td>
</tr>
<tr>
<td>---</td>
<td>----</td>
<td>----</td>
<td>----</td>
<td>----</td>
<td>-----------</td>
<td>------</td>
<td>----------</td>
</tr>
<tr>
<td>164</td>
<td>86</td>
<td>38</td>
<td>20</td>
<td></td>
<td>total</td>
<td></td>
<td></td>
</tr>
<tr>
<td>41</td>
<td>21.5</td>
<td>9.5</td>
<td>5</td>
<td></td>
<td>total/4</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

- Effects: $q_0 = 41$, $q_A = 21.5$, $q_B = 9.5$, $q_{AB} = 5$. 
Estimation of Experimental Errors

- Estimated Response:

\[ \hat{y}_i = q_0 + q_A x_{A_i} + q_B x_{B_i} + q_{AB} x_{A_i} x_{B_i} \]

Experimental Error = Measured - Estimated

\[ e_{i,j} = y_{ij} - \hat{y}_i \]

\[ = y_{ij} - q_0 - q_A x_{A_i} - q_B x_{B_i} - q_{AB} x_{A_i} x_{B_i} \]

\[ \sum_{i,j} e_{i,j} = 0 \]

Sum of Squared Errors: \[ \text{SSE} = \sum_{i=1}^{2^2} \sum_{j=1}^{r} e_{i,j}^2 \]
## Experimental Errors: Example

- **Estimated Response:**
  \[ \hat{y}_1 = q_0 - q_A - q_B + q_{AB} = 41 - 21.5 - 9.5 + 5 = 15 \]

- **Experimental errors:**
  \[ e_{11} = y_{11} - \hat{y}_1 = 15 - 15 = 0 \]

<table>
<thead>
<tr>
<th>i</th>
<th>I</th>
<th>A</th>
<th>B</th>
<th>A</th>
<th>B</th>
<th>Estimated Response</th>
<th>Measured Responses</th>
<th>Errors</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td></td>
<td>15</td>
<td>15 18 12</td>
<td>0 3 -3</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td></td>
<td>48</td>
<td>45 48 51</td>
<td>-3 0 3</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td></td>
<td>24</td>
<td>25 28 19</td>
<td>1 4 -5</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
<td>77</td>
<td>75 75 81</td>
<td>-2 -2 4</td>
</tr>
</tbody>
</table>
Allocation of Variation

- Total variation or total sum of squares:

$$\text{SST} = \sum_{i,j} (y_{ij} - \bar{y}_{..})^2$$

$$y_{ij} = q_0 + q_A x_{Ai} + q_B x_{Bi} + q_{AB} x_{Ai}x_{Bi} + e_{ij}$$

$$\sum_{i,j} (y_{ij} - \bar{y}_{..})^2 = 2^2 r q_A^2 + 2^2 r q_B^2 + 2^2 r q_{AB}^2 + \sum_{i,j} e_{ij}^2$$

$$\text{SST} = \text{SSA} + \text{SSB} + \text{SSAB} + \text{SSE}$$
Example 18.3: Memory-Cache Study

\[
\begin{align*}
SSY &= 15^2 + 18^2 + 12^2 + 45^2 + \cdots + 75^2 + 75^2 + 81^2 \\
&= 27204 \\
SS0 &= 2^2 r q_0^2 = 12 \times 41^2 = 20172 \\
SSA &= 2^2 r q_A^2 = 12 \times (21.5)^2 = 5547 \\
SSB &= 2^2 r q_B^2 = 12 \times (9.5)^2 = 1083 \\
SSAB &= 2^2 r q_{AB}^2 = 12 \times 5^2 = 300 \\
SSE &= 27204 - 2^2 \times 3(41^2 + 21.5^2 + 9.5^2 + 5^2) \\
&= 102 \\
SST &= SSY - SS0 \\
&= 27204 - 20172 = 7032
\end{align*}
\]
Example 18.3 (Cont)

\[ SSA + SSB + SSAB + SSE \]
\[ = 5547 + 1083 + 300 + 102 \]
\[ = 7032 = SST \]

Factor A explains \( \frac{5547}{7032} \) or 78.88%
Factor B explains 15.40%
Interaction AB explains 4.27%
1.45% is unexplained and is attributed to errors.
Confidence Intervals For Effects

- Effects are random variables.
- Errors $\sim N(0, \sigma_e) \Rightarrow y \sim N(\bar{y}; \sigma_e)$
  \[ q_0 = \frac{1}{2^2r} \sum_{i,j} y_{ij} \]
- $q_0 = $ Linear combination of normal variates
  $\Rightarrow q_0$ is normal with variance $\sigma_e^2/(2^2r)$

Variance of errors:

\[ s_e^2 = \frac{1}{2^2(r-1)} \sum_{i,j} e_{ij}^2 = \frac{\text{SSE}}{2^2(r-1)} \triangle \text{MSE} \]

- Denominator $= 2^2(r-1) = $ # of independent terms in SSE
  $\Rightarrow$ SSE has $2^2(r-1)$ degrees of freedom.

Estimated variance of $q_0$: $s_{q_0}^2 = s_e^2/(2^2r)$
Similarly,

\[ s_{q_A} = s_{q_B} = s_{q_{AB}} = \frac{s_e}{\sqrt{2^2 r}} \]

Confidence intervals (CI) for the effects:

\[ q_i \mp t_{[1-\alpha/2; 2^2 (r-1)]} s_{q_i} \]

CI does not include a zero ⇒ significant
Example 18.4

- For Memory-cache study: Standard deviation of errors:
  
  \[ s_e = \sqrt{\frac{SSE}{2^2(r - 1)}} = \sqrt{\frac{102}{8}} = \sqrt{12.75} = 3.57 \]

- Standard deviation of effects:
  
  \[ s_q_i = s_e / \sqrt{(2^2r)} = 3.57 / \sqrt{12} = 1.03 \]

- For 90% Confidence: \( t_{[0.95,8]} = 1.86 \)

- Confidence intervals: \( q_i \mp (1.86)(1.03) = q_i \mp 1.92 \)
  
  \( q_0 = (39.08, 42.91) \)
  
  \( q_A = (19.58, 23.41) \)
  
  \( q_B = (7.58, 11.41) \)
  
  \( q_{AB} = (3.08, 6.91) \)

- No zero crossing \( \Rightarrow \) All effects are significant.
Confidence Intervals for Contrasts

- Contrast: Linear combination with $\sum$ coefficients = 0
  $$\sum h_i q_i \text{ with } \sum h_i = 0$$
  For example, $q_A - q_B$ or $q_A + q_B - 2q_{AB}$

- Mean of $\sum h_i q_i = \sum h_i E[q_i]$ 

- Variance of $\sum h_i q_i$
  $$s^2_{\sum h_i q_i} = \frac{s^2_e \sum h_i^2}{2^2 r}$$

- For 100(1-$\alpha$)% confidence interval, use $t_{[1-\alpha/2; 2^2(r-1)]}$. 
Example 18.5

Memory-cache study

\[ u = q_A + q_B - 2q_{AB} \]

Coefficients = 0, 1, 1, and -2 \( \Rightarrow \) Contrast

Mean \( \bar{u} = 21.5 + 9.5 - 2 \times 5 = 21 \)

Variance \( s^2_u = \frac{s^2_e \times 6}{2^2 \times 3} = 6.375 \)

Standard deviation \( s_u = \sqrt{6.375} = 2.52 \)

\( t_{[0.95;8]} = 1.86 \)

90% Confidence interval for \( u \):

\[ \bar{u} \pm t s_u = 21 \pm 1.86 \times 2.52 = (16.31, 25.69) \]
Conf. Interval For Predictions

- Mean response $\hat{y}$:
  $$\hat{y} = q_0 + q_A x_A + q_B x_B + q_{AB} x_A x_B$$

- The standard deviation of the mean of $m$ responses:
  $$s_{\hat{y}_m} = s_e \left( \frac{1}{n_{\text{eff}}} + \frac{1}{m} \right)^{1/2}$$

  $$n_{\text{eff}} = \frac{\text{Effective deg of freedom}}{1 + \text{Sum of DFs of params used in } \hat{y}}$$
  $$= \frac{2^2 r}{5}$$
Conf. Interval for Predictions (Cont)

100(1-\(\alpha\))% confidence interval:

\[
\hat{y} \pm t_{[1-\alpha/2; 2^2(r-1)]} s \hat{y}_m
\]

- A single run (\(m=1\)): \(s \hat{y}_1 = s_e \left( \frac{5}{2^2r} + 1 \right)^{1/2}\)

- Population mean (\(m=\infty\)): \(s \hat{y} = s_e \left( \frac{5}{2^2r} \right)^{1/2}\)
Example 18.6: Memory-cache Study

- For \( x_A = -1 \) and \( x_B = -1 \):
  - A single confirmation experiment:
    \[
    \hat{y}_1 = q_0 - q_A - q_B + q_{AB} \\
    = 41 - 21.5 - 9.5 + 5 = 15
    \]

- Standard deviation of the prediction:
  \[
  s_{\hat{y}_1} = s_e \left( \frac{5}{2^2 r} + 1 \right)^{1/2} = 3.57 \sqrt{\frac{5}{12} + 1} = 4.25
  \]

- Using \( t_{[0.95;8]} = 1.86 \), the 90% confidence interval is:
  \[
  15 \mp 1.86 \times 4.25 = (7.09, 22.91)
  \]
Example 18.6 (Cont)

- Mean response for 5 experiments in future:

\[
\hat{y}_1 = s_e \left( \frac{5}{2^2 r} + \frac{1}{m} \right)^{1/2}
\]

\[
= 3.57 \sqrt{\frac{5}{12} + \frac{1}{5}} = 2.80
\]

- The 90% confidence interval is:

\[
15 \mp 1.86 \times 2.80 = (9.79, 20.21)
\]
Example 18.6 (Cont)

- Mean response for a large number of experiments in future:
  
  \[
  s_{\hat{y}_1} = s_e \left( \frac{5}{2^2 r} \right)^{1/2} = 3.57 \sqrt{\frac{5}{12}} = 2.30
  \]

- The 90% confidence interval is:
  
  \[
  15 \mp 1.86 \times 2.30 = (10.72, 19.28)
  \]

- Current mean response: Not for future. Use contrasts formula.
  
  \[
  s_{\hat{y}_1} = \sqrt{\frac{s_e^2 \sum h_i^2}{2^2 r}} = \sqrt{\frac{12.75 \times 4}{12}} = 2.06
  \]

- 90% confidence interval:
  
  \[
  15 \mp 1.86 \times 2.06 = (11.17, 18.83)
  \]
Updated Exercise 18.1: The following table lists measured CPU times for two processors on two workloads. Each experiment was repeated three times. Determine the effects.

<table>
<thead>
<tr>
<th>Workload</th>
<th>Processor A</th>
<th>Processor B</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>(41.16, 39.02, 42.56)</td>
<td>(65.17, 69.25, 64.23)</td>
</tr>
<tr>
<td>J</td>
<td>(53.50, 55.50, 50.50)</td>
<td>(50.08, 48.98, 47.10)</td>
</tr>
</tbody>
</table>
Assumptions

1. Errors are statistically independent.
2. Errors are additive.
3. Errors are normally distributed.
4. Errors have a constant standard deviation $\sigma_e$.
5. Effects of factors are additive
   $\Rightarrow$ observations are independent and normally distributed with constant variance.
Visual Tests

1. **Independent Errors:**
   - Scatter plot of residuals versus the predicted response $\hat{y}_i$
   - Magnitude of residuals $< \text{Magnitude of responses}/10$
     $\Rightarrow$ Ignore trends
   - Plot the residuals as a function of the experiment number
   - Trend up or down $\Rightarrow$ other factors or side effects

2. **Normally distributed errors:**
   Normal quantile-quantile plot of errors

3. **Constant Standard Deviation of Errors:**
   Scatter plot of $y$ for various levels of the factor
   Spread at one level significantly different than that at other
   $\Rightarrow$ Need transformation
Example 18.7: Memory-cache
Multiplicative Models

- Additive model:
  \[ y_{ij} = q_0 + q_A x_A + q_B x_B + q_{AB} x_A x_B + \epsilon_{ij} \]

- Not valid if effects do not add.
  E.g., execution time of workloads.
  \( i \)th processor speed= \( v_i \) instructions/second.
  \( j \)th workload Size= \( w_j \) instructions

- The two effects multiply. Logarithm \( \Rightarrow \) additive model:
  Execution Time \( y_{ij} = w_j/v_i \)
  \[ \log(y_{ij}) = \log(w_j) - \log(v_i) \]

- Correct Model:
  \[ y'_{ij} = q_0 + q_A x_A + q_B x_B + q_{AB} x_A x_B + \epsilon_{ij} \]
  Where, \( y'_{ij} = \log(y_{ij}) \)
Multiplicative Model (Cont)

- Taking an antilog of effects:
  \[ u_A = 10^{q_A}, \quad u_B = 10^{q_B}, \quad \text{and} \quad u_{AB} = 10^{q_{AB}} \]

- \( u_A \) = ratio of MIPS rating of the two processors
- \( u_B \) = ratio of the size of the two workloads.

- Antilog of additive mean \( q_0 \) \( \Rightarrow \) geometric mean

\[
\hat{y} = 10^{q_0} = (y_1 y_2 \cdots y_n)^{1/n} \quad n = 2^r
\]
Example 18.8: Execution Times

Analysis Using an Additive Model

<table>
<thead>
<tr>
<th>I</th>
<th>A</th>
<th>B</th>
<th>AB</th>
<th>y</th>
<th>Mean $\bar{y}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>(85.10, 79.50, 147.90)</td>
<td>104.170</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>(0.891, 1.047, 1.072)</td>
<td>1.003</td>
</tr>
<tr>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>(0.955, 0.933, 1.122)</td>
<td>1.003</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>(0.0148, 0.0126, 0.0118)</td>
<td>0.013</td>
</tr>
</tbody>
</table>

106.19 -104.15 -104.15 102.17 total
26.55 -26.04 -26.04 25.54 total/4

Additive model is not valid because:

- Physical consideration $\Rightarrow$ effects of workload and processors do not add. They multiply.
- Large range for $y$. $y_{\text{max}}/y_{\text{min}} = 147.90/0.0118$ or 12,534
  $\Rightarrow$ log transformation
- Taking an arithmetic mean of 104.17 and 0.013 is inappropriate.
Example 18.8 (Cont)

- The residuals are not small as compared to the response.
- The spread of residuals is large at larger value of the response.

⇒ log transformation
Example 18.8 (Cont)

- Residual distribution has a longer tail than normal
## Analysis Using Multiplicative Model

### Data After Log Transformation

<table>
<thead>
<tr>
<th>I</th>
<th>A</th>
<th>B</th>
<th>AB</th>
<th>y</th>
<th>Mean</th>
<th>$\bar{y}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>1.93</td>
<td>1.90</td>
<td>2.17</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>-0.05</td>
<td>0.02</td>
<td>0.03</td>
</tr>
<tr>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>-0.02</td>
<td>-0.03</td>
<td>0.05</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-1.83</td>
<td>-1.90</td>
<td>-1.93</td>
</tr>
<tr>
<td></td>
<td>0.11</td>
<td>-3.89</td>
<td>-3.89</td>
<td>0.11</td>
<td></td>
<td>total</td>
</tr>
<tr>
<td></td>
<td>0.03</td>
<td>-0.97</td>
<td>-0.97</td>
<td>0.03</td>
<td></td>
<td>total/4</td>
</tr>
</tbody>
</table>
Variation Explained by the Two Models

<table>
<thead>
<tr>
<th>Factor</th>
<th>Effect</th>
<th>% Var.</th>
<th>Conf. Interval</th>
<th>Effect</th>
<th>% Var.</th>
<th>Conf. Interval</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>26.55</td>
<td></td>
<td>(16.35, 36.74)</td>
<td>0.03</td>
<td></td>
<td>(-0.02, 0.07)†</td>
</tr>
<tr>
<td>A</td>
<td>-26.04</td>
<td>30.1%</td>
<td>(-36.23, -15.84)</td>
<td>-0.97</td>
<td>49.9%</td>
<td>(-1.02, -0.93)</td>
</tr>
<tr>
<td>B</td>
<td>-26.04</td>
<td>30.1%</td>
<td>(-36.23, -15.84)</td>
<td>-0.97</td>
<td>49.9%</td>
<td>(-1.02, -0.93)</td>
</tr>
<tr>
<td>AB</td>
<td>25.54</td>
<td>29.0%</td>
<td>(15.35, 35.74)</td>
<td>0.03</td>
<td>0.0%</td>
<td>(-0.02, 0.07)†</td>
</tr>
<tr>
<td>e</td>
<td>10.8%</td>
<td></td>
<td></td>
<td></td>
<td>0.2%</td>
<td></td>
</tr>
</tbody>
</table>

† ⇒ Not Significant

- With multiplicative model:
  - Interaction is almost zero.
  - Unexplained variation is only 0.2%
Visual Tests

- **Conclusion**: Multiplicative model is better than the additive model.
Interpretation of Results

\[ \log(y) = q_0 + q_A x_A + q_B x_B + q_{AB} x_A x_B + \epsilon \]

\[ \Rightarrow y = 10^{q_0} 10^{q_A x_A} 10^{q_B x_B} 10^{q_{AB} x_A x_B} 10^{\epsilon} \]
\[ = 10^{0.03} 10^{-0.97 x_A} 10^{-0.97 x_B} 10^{0.03 x_A x_B} 10^{\epsilon} \]
\[ = 1.07 \times 0.107^{x_A} \times 0.107^{x_B} \times 1.07^{x_A x_B} 10^{\epsilon} \]

- The time for an average processor on an average benchmark is 1.07.
- The time on processor A_1 is 9.35 times (0.107^{-1}) that on an average processor. The time on A_2 is 1/9.35 (0.107^1) of that on an average processor.
- MIPS rate for A_2 is 87.4 times that of A_1.
- Benchmark B_1 executes 87.4 times more instructions than B_2.
- The interaction is negligible.

\[ \Rightarrow \text{Results apply to all benchmarks and processors.} \]
Transformation Considerations

- $y_{\text{max}}/y_{\text{min}}$ small $\Rightarrow$ Multiplicative model results similar to additive model.
- Many other transformations possible.
- Box-Cox family of transformations:
  
  $$w = \begin{cases} \frac{y^a-1}{a g^a-1}, & a \neq 0 \\ (\ln y) g, & a = 0 \end{cases}$$

- Where $g$ is the geometric mean of the responses:
  
  $$g = (y_1 y_2 \cdots y_n)^{1/n}$$

- $w$ has the same units as $y$.
- $a$ can have any real value, positive, negative, or zero.
- Plot SSE as a function of $a \Rightarrow$ optimal $a$
- Knowledge about the system behavior should always take precedence over statistical considerations.
General $2^{kr}$ Factorial Design

- Model:
  \[ y_{ij} = q_0 + q_A x_A i + q_B x_B i + q_{AB} x_A i x_B i + \cdots + e_{ij} \]

- Parameter estimation:
  \[ q_j = \frac{1}{2^k} \sum_{i=1}^{2^k} S_{ij} \bar{y}_i \]
  \[ S_{ij} = (i,j)\text{th entry in the sign table.} \]

- Sum of squares:
  \[ \text{SSY} = \sum_{i=1}^{2^k} \sum_{j=1}^{r} y_{ij}^2 \]
  \[ \text{SS0} = 2^k r q_0^2 \]
  \[ \text{SST} = \text{SSY} - \text{SS0} \]
  \[ \text{SS}_j = 2^k r q_j^2 \quad j = 1, 2, \ldots, 2^k - 1 \]
  \[ \text{SSE} = \text{SST} - \sum_{j=1}^{2^k-1} \text{SS}_j \]

  \[ \text{SST} = \text{SSY} - \text{SS0} = \sum_{j=1}^{2^k} \text{SS}_j + \text{SSE} \]
  \[ 2^k r - 1 = 2^k r - 1 = \sum_{j=1}^{2^k} 1 + 2^k (r - 1) \]
General $2^kr$ Factorial Design (Cont)

- Percentage of y's variation explained by $j$th effect:
  \[ (SS_j / SST) \times 100\% \]

- Standard deviation of errors:
  \[ s_e = \sqrt{\frac{SSE}{2^k(r-1)}} \]

- Standard deviation of effects:
  \[ s_{q_0} = s_{q_A} = s_{q_B} = s_{q_{AB}} = s_e / \sqrt{2^k r} \]

- Variance of contrast $\sum h_i q_i$, where $\sum h_i = 0$ is:
  \[ s_{\sum h_i q_i}^2 = (s_e^2 \sum h_i^2) / 2^k r \]
General $2^k r$ Factorial Design (Cont)

- Standard deviation of the mean of $m$ future responses:
  \[ s_{\hat{y}_p} = s_e \left( \frac{1 + 2^k}{2^k r} + \frac{1}{m} \right)^{1/2} \]

- Confidence intervals are calculated using $t_{[1-\alpha/2;2^k(r-1)]}$.

- Modeling assumptions:
  - Errors are IID normal variates with zero mean.
  - Errors have the same variance for all values of the predictors.
  - Effects and errors are additive.
Visual Tests for $2^k r$ Designs

- The scatter plot of errors versus predicted responses should not have any trend.
- The normal quantile-quantile plot of errors should be linear.
- Spread of y values in all experiments should be comparable.
Example 18.9: A $2^33$ Design

<table>
<thead>
<tr>
<th>I</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>A B</th>
<th>A C</th>
<th>B C</th>
<th>A B C</th>
<th>y</th>
<th>Mean $\bar{y}$</th>
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<td>1</td>
<td>80</td>
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</table>

319  67  43  155  23  19  15  -1  total
39.87 8.375 5.375 19.37 2.875 2.375 1.875 -0.125 total/8
Example 18.9 (Cont)

- Sum of Squares:

<table>
<thead>
<tr>
<th>Component</th>
<th>Sum of Squares</th>
<th>Percent Variation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>4.9E4</td>
<td></td>
</tr>
<tr>
<td>$\bar{y}$</td>
<td>3.8E4</td>
<td></td>
</tr>
<tr>
<td>$y-\bar{y}$</td>
<td>1.1E4</td>
<td>100.00%</td>
</tr>
<tr>
<td>A</td>
<td>1683.0</td>
<td>14.06%</td>
</tr>
<tr>
<td>B</td>
<td>693.3</td>
<td>5.79%</td>
</tr>
<tr>
<td>C</td>
<td>9009.0</td>
<td>75.27%</td>
</tr>
<tr>
<td>AB</td>
<td>198.3</td>
<td>1.66%</td>
</tr>
<tr>
<td>AC</td>
<td>135.4</td>
<td>1.13%</td>
</tr>
<tr>
<td>BC</td>
<td>84.4</td>
<td>0.70%</td>
</tr>
<tr>
<td>ABC</td>
<td>0.4</td>
<td>0.00%</td>
</tr>
<tr>
<td>Errors</td>
<td>164.0</td>
<td>1.37%</td>
</tr>
</tbody>
</table>
Example 18.9 (Cont)

- The errors have $2^3(3-1)$ or 16 degrees of freedom. Standard deviation of errors:

$$s_e = \sqrt{\frac{\text{SSE}}{2^k(r - 1)}} = \sqrt{\frac{164}{16}} = 3.20$$

- Standard deviation of effects:

$$s_{qi} = s_e / \sqrt{2^33} = 3.20 / \sqrt{24} = 0.654$$
% Variation:

<table>
<thead>
<tr>
<th>Component</th>
<th>Sum of Squares</th>
<th>Percent Variation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
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<td></td>
</tr>
<tr>
<td>$\bar{y}$</td>
<td>3.8E4</td>
<td></td>
</tr>
<tr>
<td>$y-\bar{y}$</td>
<td>1.1E4</td>
<td>100.00%</td>
</tr>
<tr>
<td>A</td>
<td>1683.0</td>
<td>14.06%</td>
</tr>
<tr>
<td>B</td>
<td>693.3</td>
<td>5.79%</td>
</tr>
<tr>
<td>C</td>
<td>9009.0</td>
<td>75.27%</td>
</tr>
<tr>
<td>AB</td>
<td>198.3</td>
<td>1.66%</td>
</tr>
<tr>
<td>AC</td>
<td>135.4</td>
<td>1.13%</td>
</tr>
<tr>
<td>BC</td>
<td>84.4</td>
<td>0.70%</td>
</tr>
<tr>
<td>ABC</td>
<td>0.4</td>
<td>0.00%</td>
</tr>
<tr>
<td>Errors</td>
<td>164.0</td>
<td>1.37%</td>
</tr>
</tbody>
</table>
Example 18.9 (Cont)

- \( t_{[0.90, 16]} = 1.337 \)
- 80% confidence intervals for parameters: \( q_i \pm (1.337)(0.654) = q_i \pm 0.874 \)

- \( q_0 = (39.00, 40.74) \)
- \( q_A = (7.50, 9.25) \)
- \( q_B = (4.50, 6.25) \)
- \( q_C = (18.50, 20.24) \)
- \( q_{AB} = (2.00, 3.75) \)
- \( q_{AC} = (1.50, 3.25) \)
- \( q_{BC} = (1.00, 2.75) \)
- \( q_{ABC} = (-1.00, 0.75) \)

- All effects except \( q_{ABC} \) are significant.
Example 18.9 (Cont)

- For a single confirmation experiment \((m = 1)\)
  
  With \(A = B = C = -1:\)
  
  \[
  \hat{y} = 14
  \]
  
  \[
  s_{\hat{y}} = s_e \left( \frac{9}{2k_r} + \frac{1}{m} \right)^{1/2}
  \]
  
  \[
  = 3.2 \left( \frac{9}{24} + 1 \right)^{1/2}
  \]
  
  \[
  = 3.75
  \]

- 80% confidence interval:
  
  \[
  14 \pm 1.337 \times 3.75 = 14 \pm 5.02 = (8.98, 19.02)
  \]
Importance vs. Significance

- Important = Parameters or models that explain a high percent of variation
- Significant = Parameters or models whose confidence interval does not include zero

**Important**
- Run more experiments

**Significant**
- Ignore the parameter and see if the model is still significant

Good
# Case Study 18.1: Garbage collection

## Factors and Levels

<table>
<thead>
<tr>
<th>Variable</th>
<th>Factor</th>
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<tbody>
<tr>
<td>A</td>
<td>Workload</td>
<td>Single Task</td>
<td>Several parallel tasks</td>
</tr>
<tr>
<td>B</td>
<td>Compiler</td>
<td>Simple</td>
<td>Deallocating</td>
</tr>
<tr>
<td>C</td>
<td>Limbo List</td>
<td>Enabled</td>
<td>Disabled</td>
</tr>
<tr>
<td>D</td>
<td>Chunk Size</td>
<td>4K bytes</td>
<td>16K bytes</td>
</tr>
<tr>
<td>I</td>
<td>A</td>
<td>B</td>
<td>C</td>
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<tr>
<td>---</td>
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<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

2695.67  -1344.33  4.33  9.00  1667.00
total

168.48  -84.02  0.27  0.56  104.19
total/16
### Case Study 18.1 (Cont)

<table>
<thead>
<tr>
<th>Factor</th>
<th>Effect</th>
<th>% Variation</th>
<th>Conf. Interval</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>168.48</td>
<td></td>
<td>(168.386, 168.573)</td>
</tr>
<tr>
<td>A</td>
<td>-84.02</td>
<td>34.4%</td>
<td>(-84.114, -83.927)</td>
</tr>
<tr>
<td>B</td>
<td>0.27</td>
<td>0.0%</td>
<td>(0.177, 0.364)</td>
</tr>
<tr>
<td>C</td>
<td>0.56</td>
<td>0.0%</td>
<td>(0.469, 0.656)</td>
</tr>
<tr>
<td>D</td>
<td>104.19</td>
<td>52.8%</td>
<td>(104.094, 104.281)</td>
</tr>
<tr>
<td>AB</td>
<td>-0.23</td>
<td>0.0%</td>
<td>(-0.323, -0.136)</td>
</tr>
<tr>
<td>AC</td>
<td>0.56</td>
<td>0.0%</td>
<td>(0.469, 0.656)</td>
</tr>
<tr>
<td>AD</td>
<td>-51.31</td>
<td>12.8%</td>
<td>(-51.406, -51.219)</td>
</tr>
<tr>
<td>BC</td>
<td>0.02</td>
<td>0.0%</td>
<td>(-0.073, 0.114)</td>
</tr>
<tr>
<td>BD</td>
<td>0.23</td>
<td>0.0%</td>
<td>(0.136, 0.323)</td>
</tr>
<tr>
<td>CD</td>
<td>0.44</td>
<td>0.0%</td>
<td>(0.344, 0.531)</td>
</tr>
<tr>
<td>ABC</td>
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<td>0.0%</td>
<td>(-0.073, 0.114)</td>
</tr>
<tr>
<td>ABD</td>
<td>-0.27</td>
<td>0.0%</td>
<td>(-0.364, -0.177)</td>
</tr>
<tr>
<td>ACD</td>
<td>0.44</td>
<td>0.0%</td>
<td>(0.344, 0.531)</td>
</tr>
<tr>
<td>BCD</td>
<td>-0.02</td>
<td>0.0%</td>
<td>(-0.114, 0.073)</td>
</tr>
<tr>
<td>ABCD</td>
<td>-0.02</td>
<td>0.0%</td>
<td>(-0.114, 0.073)</td>
</tr>
</tbody>
</table>

† ⇒ Not Significant
Case Study 18.1: Conclusions

- Most of the variation is explained by factors A (Workload), D (Chunk size), and the interaction AD between the two.

- The variation due to experimental error is small
  ⇒ Several effects that explain less than 0.05% of variation (listed as 0.0%) are statistically significant.

- Only effects A, D, and AD are both practically significant and statistically significant.
Summary

- Replications allow estimation of measurement errors
  ⇒ Confidence Intervals of parameters
  ⇒ Confidence Intervals of predicted responses
- Allocation of variation is proportional to square of effects
- Multiplicative models are appropriate if the factors multiply
- Visual tests for independence normal errors
Homework 18B

Updated Exercise 18.1: For the data of Homework 18A, determine percentage of variation explained, find confidence intervals of the effects, and conduct visual tests.