Random Variate Generation

Raj Jain
Washington University in Saint Louis
Saint Louis, MO 63130
Jain@cse.wustl.edu

Audio/Video recordings of this lecture are available at:
http://www.cse.wustl.edu/~jain/cse567-08/
Overview

1. Inverse transformation
2. Rejection
3. Composition
4. Convolution
5. Characterization
Random-Variate Generation

- General Techniques
- Only a few techniques may apply to a particular distribution
- Look up the distribution in Chapter 29
Inverse Transformation

- Used when $F^{-1}$ can be determined either analytically or empirically.

$$u = F(x) \sim U(0, 1)$$

$$x = F^{-1}(u)$$
Proof

Let \( y = g(x) \), so that \( x = g^{-1}(y) \).

\[
F_Y(y) = P(Y \leq y) = P(x \leq g^{-1}(y)) = F_X(g^{-1}(y))
\]

If \( g(x) = F(x) \), or \( y = F(x) \)

\[
F(y) = F(F^{-1}(y)) = y
\]

And:

\[
f(y) = dF/dy = 1
\]

That is, \( y \) is uniformly distributed between 0 and 1.
Example 28.1

- For exponential variates:

  The pdf \( f(x) = \lambda e^{-\lambda x} \)
  The CDF \( F(x) = 1 - e^{-\lambda x} = u \) or, \( x = -\frac{1}{\lambda} \ln(1 - u) \)

- If \( u \) is U(0,1), 1-\( u \) is also U(0,1)

- Thus, exponential variables can be generated by:

  \[ x = -\frac{1}{\lambda} \ln(u) \]
Example 28.2

- The packet sizes (trimodal) probabilities:

<table>
<thead>
<tr>
<th>Size</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>64 Bytes</td>
<td>0.7</td>
</tr>
<tr>
<td>128 Bytes</td>
<td>0.1</td>
</tr>
<tr>
<td>512 Bytes</td>
<td>0.2</td>
</tr>
</tbody>
</table>

- The CDF for this distribution is:

\[
F(x) = \begin{cases} 
0.0 & 0 \leq x < 64 \\
0.7 & 64 \leq x < 128 \\
0.8 & 128 \leq x < 512 \\
1.0 & 512 \leq x 
\end{cases}
\]
Example 28.2 (Cont)

- The inverse function is:

\[ F^{-1}(u) = \begin{cases} 
64 & 0 < u \leq 0.7 \\
128 & 0.7 < u \leq 0.8 \\
512 & 0.8 < u \leq 1 
\end{cases} \]

Generate \( u \sim U(0, 1) \)

- \( u \leq 0.7 \Rightarrow \text{Size} = 64 \)
- \( 0.7 < u \leq 0.8 \Rightarrow \text{size} = 128 \)
- \( 0.8 < u \Rightarrow \text{size} = 512 \)

- Note: CDF is continuous from the right
  \( \Rightarrow \) the value on the right of the discontinuity is used
  \( \Rightarrow \) The inverse function is continuous from the left
  \( \Rightarrow u=0.7 \Rightarrow x=64 \)
### Applications of the Inverse-Transformation Technique

<table>
<thead>
<tr>
<th>Distribution</th>
<th>CDF $F'(x)$</th>
<th>Inverse</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exponential</td>
<td>$1 - e^{-x/a}$</td>
<td>$-a \ln(u)$</td>
</tr>
<tr>
<td>Extreme value</td>
<td>$1 - e^{-e^{x-a/b}}$</td>
<td>$a + b \ln \ln u$</td>
</tr>
<tr>
<td>Geometric</td>
<td>$1 - (1 - p)^x$</td>
<td>$\left[ \frac{\ln(u)}{\ln(1-p)} \right]$</td>
</tr>
<tr>
<td>Logistic</td>
<td>$1 - \frac{1}{1+e^{-x/\mu}}$</td>
<td>$\mu - b \ln\left(\frac{1}{u} - 1\right)$</td>
</tr>
<tr>
<td>Pareto</td>
<td>$1 - x^{-a}$</td>
<td>$1/u^{1/a}$</td>
</tr>
<tr>
<td>Weibull</td>
<td>$1 - e^{(x/\alpha)^b}$</td>
<td>$\alpha (\ln u)^{1/b}$</td>
</tr>
</tbody>
</table>
Rejection

- Can be used if a pdf $g(x)$ exists such that $c \, g(x)$ majorizes the pdf $f(x) \Rightarrow c \, g(x) \geq f(x) \ \forall \ x$

- Steps:
  1. Generate $x$ with pdf $g(x)$.
  2. Generate $y$ uniform on $[0, cg(x)]$.
  3. If $y \leq f(x)$, then output $x$ and return.
     Otherwise, repeat from step 1.
     \[ \Rightarrow \text{Continue rejecting the random variates } x \text{ and } y \text{ until } y \geq f(x) \]

- Efficiency = how closely $c \, g(x)$ envelopes $f(x)$
  Large area between $c \, g(x)$ and $f(x)$ \Rightarrow Large percentage of $(x, y)$ generated in steps 1 and 2 are rejected

- If generation of $g(x)$ is complex, this method may not be efficient.
Example 28.2

- Beta(2.4) density function:
  \[ f(x) = 20x(1 - x)^3 \quad 0 \leq x \leq 1 \]
  \[ c=2.11 \quad \text{and} \quad g(x) = 1 \quad 0 \leq x \leq 1 \]

- Bounded inside a rectangle of height 2.11
  \[ \Rightarrow \text{Steps:} \]
  - Generate x uniform on [0, 1].
  - Generate y uniform on [0, 2.11].
  - If \( y \leq 20 x(1-x)^3 \), then output x and return. Otherwise repeat from step 1.
Composition

- Can be used if CDF $F(x) = \text{Weighted sum of n other CDFs.}$
  $$F(x) = \sum_{i=1}^{n} p_i F_i(x)$$

- Here, $p_i \geq 0$, $\sum_{i=1}^{n} p_i = 1$, and $F_i$'s are distribution functions.

- $n$ CDFs are composed together to form the desired CDF
  Hence, the name of the technique.

- The desired CDF is decomposed into several other CDFs
  ⇒ Also called decomposition.

- Can also be used if the pdf $f(x)$ is a weighted sum of $n$ other pdfs:
  $$f(x) = \sum_{i=1}^{n} p_i f_i(x)$$
Steps:

- Generate a random integer $I$ such that:
  \[ P(I = i) = p_i \]
- This can easily be done using the inverse-transformation method.
- Generate $x$ with the $i$th pdf $f_i(x)$ and return.
Example 28.4

- pdf: \( f(x) = \frac{1}{2a} e^{-|x|/a} \)
- Composition of two exponential pdf's
- Generate
  \[ u_1 \sim U(0, 1) \]
  \[ u_2 \sim U(0, 1) \]
- If \( u_1 < 0.5 \), return; otherwise return \( x = a \ln u_2 \).
- Inverse transformation better for Laplace
Convolution

- Sum of \( n \) variables: \( x = y_1 + y_2 + \cdots + y_n \)
- Generate \( n \) random variate \( y_i \)'s and sum
- For sums of two variables, pdf of \( x = \) convolution of pdfs of \( y_1 \) and \( y_2 \). Hence the name
- Although no convolution in generation
- If pdf or CDF = Sum \( \Rightarrow \) Composition
- Variable \( x = \) Sum \( \Rightarrow \) Convolution

\[
f \ast g(t) = \int f(\tau)g(t-\tau)\,d\tau
\]
Convolutions: Examples

- Erlang-k = \( \sum_{i=1}^{k} \) Exponential_i
- Binomial(n, p) = \( \sum_{i=1}^{n} \) Bernoulli(p)
  \( \Rightarrow \) Generated \( n \) U(0,1), return the number of RNs less than \( p \)
- \( \chi^2(\nu) = \sum_{i=1}^{\nu} N(0,1)^2 \)
- \( \Gamma(a, b_1) + \Gamma(a, b_2) = \Gamma(a, b_1 + b_2) \)
  \( \Rightarrow \) Non-integer value of \( b = \) integer + fraction
- \( \sum_{i=1}^{n} \) Any = Normal \( \Rightarrow \) \( \sum U(0,1) = \) Normal
- \( \sum_{i=1}^{m} \) Geometric = Pascal
- \( \sum_{i=1}^{2} \) Uniform = Triangular
Characterization

- Use special characteristics of distributions ⇒ characterization
- Exponential inter-arrival times ⇒ Poisson number of arrivals
  ⇒ Continuously generate exponential variates until their sum exceeds T and return the number of variates generated as the Poisson variate.
- The $a^{th}$ smallest number in a sequence of $a+b+1$ U(0,1) uniform variates has a $\beta(a, b)$ distribution.
- The ratio of two unit normal variates is a Cauchy(0, 1) variate.
- A chi-square variate with even degrees of freedom $\chi^2(\nu)$ is the same as a gamma variate $\gamma(2,\nu/2)$.
- If $x_1$ and $x_2$ are two gamma variates $\gamma(a,b)$ and $\gamma(a,c)$, respectively, the ratio $x_1/(x_1+x_2)$ is a beta variate $\beta(b,c)$.
- If $x$ is a unit normal variate, $e^{\mu+\sigma x}$ is a lognormal($\mu, \sigma$) variate.
Is CDF invertible?

Yes

Use inversion

Is CDF a sum of other CDFs?

Yes

Use composition

Is pdf a sum of other pdfs?

Yes

Use Composition
Summary (Cont)

- **Is the variate a sum of other variates?**
  - Yes: Use convolution
  - No: Continue

- **Is the variate related to other variates?**
  - Yes: Use characterization
  - No: Continue

- **Does a majorizing function exist?**
  - Yes: Use rejection
  - No: Use empirical inversion
Exercise 28.1

A random variate has the following triangular density:

\[ f(x) = \min(x, 2 - x) \quad 0 \leq x \leq 2 \]

Develop algorithms to generate this variate using each of the following methods:

a. Inverse-transformation
b. Rejection
c. Composition
d. Convolution
Homework 28

- A random variate has the following triangular density:
  \[ f(x) = \frac{1}{16} \min(x, 8 - x) \quad 0 \leq x \leq 8 \]

- Develop algorithms to generate this variate using each of the following methods:
  a. Inverse-transformation
  b. Rejection
  c. Composition
  d. Convolution