99-0045

# Throughput Fairness Index: An Explaination

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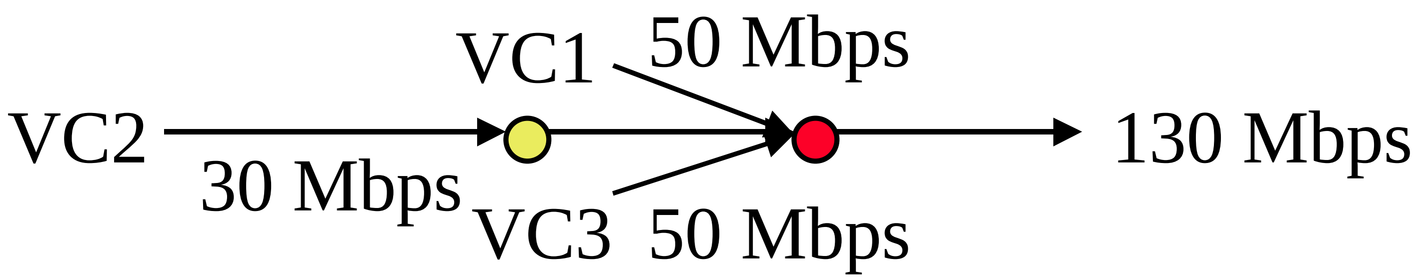
- Index of fairness
- Why is it better than others?

References:

[1] R. Jain, W. Hawe, D. Chiu, "A Quantitative measure of fairness and discrimination for resource allocation in Shared Computer Systems," DEC-TR-301, September 26, 1984, http://www.cis.ohio-state.edu/~jain/papers/fairness.htm

# Fairness




- Simple Definition: Equal share of bottleneck
  Problem: Some VC's may be bottlenecked elsewhere
- Solution: Use any of the TM4.0 fairness criteria to define optimal allocations and use a fairness index to quantify the fairness.
- Question: A scheme gives 50, 30, 50 Mbps
  when the optimal is 50, 10, 10 Mbps
  How fair is it? 67% ? 90%?

# Proposal

- Measured Throughput: $(T_1, T_2, ..., T_n)$
- Use any criterion (e.g., max-min optimality) to find the Fair Throughput $(O_1, O_2, ..., O_n)$
- Normalized Throughput: $x_i = T_i/O_i$

$$\text{Fairness Index} = \frac{(\Sigma x_i)^2}{n\Sigma x_i^2}$$

**Example**: 50/50, 30/10, 50/10 $\Rightarrow$ 1, 3, 5

$$\text{Fairness Index} = \frac{(1+3+5)^2}{3(1^2+3^2+5^2)} = \frac{9^2}{3(1+9+25)} = 0.81$$

# Other Fairness Indices

- Variance:
  Mean $\mu = (1+3+5)/3 = 3$
  Variance $\sigma^2 = 1/(n-1)\Sigma(x_i-\mu)^2 = 4$
- Coefficient of Variation: Standard deviation $\sigma = 2$
  COV $= \sigma/\mu = 0.667$
- Min-Max Ratio: $\text{Min}\{x_i\}/\text{Max}\{x_i\} = 1/5 = 0.2$
- Find the normalized distance from the optimal

$$\text{Norm. Dist.} = \frac{[\Sigma (T_i-O_i)^2]^{1/2}}{[\Sigma O_i^2]^{1/2}}$$

$$= \frac{[(50-50)^2+(30-10)^2+(50-10)^2]^{1/2}}{[50^2+10^2+10^2]^{1/2}} = 0.86$$

# Fairness Index: Properties

- Applicable for any number of VCs, even n=2
  Strictly speaking, variance not defined for small n.
- Scale independent.
  Variance (Throughput) = 10 $Mbps^2$ = $10^7$ $kbps^2$
  Standard deviation (Throughput) = 10 Mbps = $10^4$ kbps
- Bounded between 0 and 1 or 0 and 100%
  Variance, standard deviation, and Relative distance are not bounded.
- Direct relationship: Higher index ⇒ More Fair
  Higher variance ⇒ Less fair
- Continuous. Min/max is not continuous.

# Fairness Index: Properties

**Intuitive:**

- For (1, 0, 1) Index = 2/3
- For $x_i$ = 1, i=1,2,3,...,k
  = 0 otherwise
  Fairness Index = k/n
- If 80% of the users are treated fairly and 20% are starved, index = 80%
- If y% of the users are treated fairly and (100-y)% are starved, Fairness index = y%

# Relationship to Other Indices

- Fairness Index = $E[x]^2/E[x^2] = 1/(1+COV^2)$
  Transformation
  - Makes index bounded between 0 and 1,
  - Gives a direct positive relationship between the index and fairness
  - Makes it intuitive: y% if fair to y% only.

# User Perception of Fairness

- Fairness Index = $(\Sigma x_i)^2/(n\Sigma x_i^2)$
  $= (1/n)\, \Sigma x_i/x_f$
  Where $x_f = \Sigma x_i^2/\Sigma x_i$ = Fair allocation mark
- $i^{th}$ User perception of fairness = $x_i/x_f$

**Example**: 2 Mbps to first 10 users, 0 to other 90 users

$x_i = 2$, i=1,2,…,10

$x_i = 0$, i=11,12,…,100

Fair Allocation Mark $x_f = 2$

First 10 users are 100% happy and other 90 users are 0% happy. Average fairness = 10%

# Properties of Fairness Index

1. If $\delta x$ resource is taken away from kth user and given to $j^{th}$ user, the fairness
   - Increases iff $\delta x < x_k - x_j$
   - Remains the same iff $\delta x = x_k - x_j$
   - Decreases iff $\delta x > x_k - x_j$

2a. If each user is given an additional resource $\delta x$, the index goes up if it is less than 1.

$$f(x_1+c, x_2+c, \ldots, x_n+c) \geq f(x_1, x_2, \ldots, x_n)$$

2b. If a single user j is given extra $\delta x$, the index goes up iff j is a discriminated user (and vice versa):

$$f(x_1, x_2, \ldots, x_j+\delta x, \ldots, x_n) > f(x_1, x_2, \ldots, x_n) \text{ iff } x_j < x_f$$

# Properties (Cont)

3. If we vary a single user's allocation, the fairness reaches maximum when $x_j = x_g$ where $x_g$ is the fair allocation mark for the remaining n-1 users

$$x_g = \Sigma_{i \neq j} x_i^2 / \Sigma_{i \neq j} x_i$$

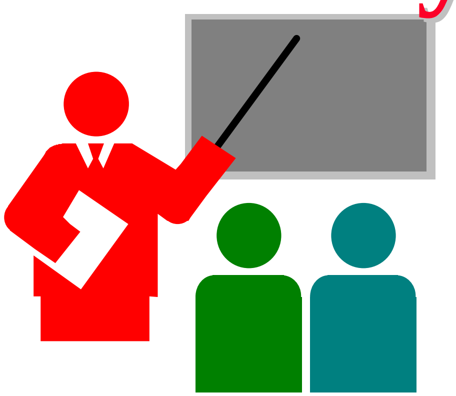
# Other Similar Functions

- $F(x) = (\Sigma x_i)^k/(n\Sigma x_i^k)$
- If you starve 90% of the users and give all resources to 10% of the users:

$$(\Sigma x_i)^k/(n\Sigma x_i^k) = 0.1^{k-1}$$

The index is 0.1 or 10% only for k=2.

# Summary



$$\text{Fairness Index} = \frac{(\Sigma x_i)^2}{n \Sigma x_i^2}$$

- Independent of scale
- Continuous
- Applies to any number of users
- Bounded between 0 and 1
- Has a intuitive relationship with user perception