EE464
Transmission Line Equations (Dally/Poulton Notation)

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Transmission Line Equations

$$V_f(t-\frac{x}{v}) \text{ and } V_r(t+\frac{x}{v})$$

are two traveling waves with V_f propagating in the positive x direction and V_r propagating in the negative x direction. Both move with velocity v.

$$V = V_f + V_r \qquad Z_0 = \sqrt{\frac{L}{C}} \qquad v = \frac{1}{\sqrt{LC}}$$

$$I = I_f + I_r \qquad L = \frac{Z_0}{v} \qquad C = \frac{1}{Z_0 v} \qquad L = \frac{1}{C \cdot v^2}$$

$$V_f = Z_0 I_f \qquad K_{rT} = \frac{Z_T - Z_0}{Z_T + Z_0}$$

$$V_r = -Z_0 I_r \qquad K_{rS} = \frac{Z_S - Z_0}{Z_S + Z_0}$$

Superposition (if linear)

General Equations

$$L \cdot C = 1/v^2 = \mu \cdot \varepsilon$$

$$L/m = 1.112E-17 \cdot \varepsilon_r /(C/m)$$

$$Z_0 = \sqrt{L/C} = 377/\sqrt{\varepsilon_r} \;\; \text{per square}$$

$$v = \sqrt{1/CL} = (3E10/\sqrt{\varepsilon_r})cm/\sec$$

$$\text{Or} \quad (1 \cdot \sqrt{\varepsilon_r})ns/ft \qquad (83 \cdot \sqrt{\varepsilon_r})ps/in \qquad (33 \cdot \sqrt{\varepsilon_r})ps/cm$$

Transmission Line Equations

$$V_{fl}(t) = V_{f0}(t - \frac{l}{v})$$

$$V_{r0}(t) = V_{rl}(t + \frac{l}{v})$$

$$V_{f0}(t) = V_S(t)\frac{Z_0}{Z_0 + Z_S} + V_{r0}(t)\frac{Z_S - Z_0}{Z_S + Z_0}$$

$$V_{rl}(t) = V_T(t)\frac{Z_0}{Z_T + Z_0} + V_{fl}(t)\frac{Z_T - Z_0}{Z_T + Z_0}$$

$$V_l = 2V_{fl}\frac{Z_T}{Z_T + Z_0} + V_T\frac{Z_0}{Z_0 + Z_T}$$

$$V_0 = V_S\frac{Z_0}{Z_0 + Z_S} + 2V_{r0}\frac{Z_S}{Z_S + Z_0}$$

General Equations ($V_T = 0$), Use Superposition for Multiple Sources

$$V_{f0}(t) = V_{r0}(t)K_{rS} + V_S(t)\frac{Z_0}{Z_0 + Z_S} \qquad 1)$$

$$V_{r0}(t) = V_{f0}(t - \frac{2l}{v})K_{rT} \qquad 2)$$

2) into 1),

$$V_{f0}(t) = V_{f0}(t - \frac{2l}{v})K_{rT} \cdot K_{rS} + V_S(t)\frac{Z_0}{Z_0 + Z_S} \qquad 3)$$

At x,

$$\begin{aligned} V_{fx}(t) &= V_{f0}(t - \frac{x}{v}) \\ &= V_{f0}(t - \frac{2l}{v} - \frac{x}{v})K_{rT} \cdot K_{rS} + V_S(t - \frac{x}{v})\frac{Z_0}{Z_0 + Z_S} \qquad 4) \end{aligned}$$

From 2) & 3),

$$\begin{aligned} V_{rx}(t) &= V_{f0}(t - \frac{4l}{v} + \frac{x}{v})K_{rT}^2 \cdot K_{rS} \\ &+ V_S(t - \frac{2l}{v} + \frac{x}{v})K_{rT} \cdot \frac{Z_0}{Z_0 + Z_S} \qquad 5) \end{aligned}$$

Equations Continued

Repeat 3) into 4), 3) into 5),

$$V_x(t)=\frac{Z_0}{Z_0+Z_S}\sum_{n=0}^{\infty}\left\{V_S(t-\underset{\underset{V_f}{\uparrow}}{\frac{x}{v}}-\frac{2nl}{v})K_{rS}^{n}\cdot K_{rT}^{n}+V_S(t-\underset{\underset{V_r}{\uparrow}}{\frac{2(n+1)l}{v}}+\frac{x}{v})K_{rT}^{n+1}\cdot \underset{\underset{0^0=1}{\uparrow}}{K_{rS}^{n}}\right\}$$

At $x = 0$,

$$V_0(t)=\frac{Z_0}{Z_0+Z_S}\left[\underset{\underset{V_f}{\uparrow}}{V_S(t)}+\sum_{n=1}^{\infty}V_S(t-\frac{2nl}{v})K_{rT}^{n}(\underset{\underset{V_r}{\uparrow}}{K_{rS}^{n-1}}+\underset{\underset{V_f}{\uparrow}}{K_{rS}^{n}})\right]$$

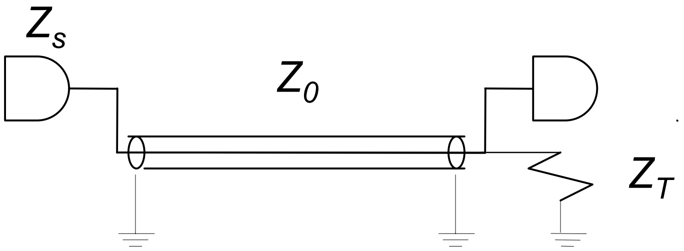
At $x = l$,

$$V_l(t)=\frac{Z_0}{Z_0+Z_S}(\underset{\underset{V_f}{\uparrow}}{1}+\underset{\underset{V_r}{\uparrow}}{K_{rT}})\sum_{n=0}^{\infty}V_S(t-\frac{2nl}{v}-\frac{l}{v})\underset{\underset{0^0=1}{\uparrow}}{K_{rT}^{n}}\cdot K_{rS}^{n}$$

Now can find V_0, V_l at any time.

Example Driving T-Line

Apply to problem,



$$V_S(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

$$V_l(t) = \frac{Z_0}{Z_0 + Z_S}(1 + K_{rT}) \sum_{n=0}^{\infty} \underbrace{V_S(t - \frac{2nl}{v} - \frac{l}{v})} K_{rT}^n \cdot K_{rS}^n$$

$$= \begin{cases} 1 & t - \frac{(2n+1)l}{v} > 0 \\ 0 & t - \frac{(2n+1)l}{v} < 0 \end{cases}$$

$$= \frac{Z_0}{Z_0 + Z_S}(1 + K_{rT}) \sum_{n=0}^{N} K_{rT}^n \cdot K_{rS}^n \qquad \text{for} \quad \frac{(2N+1)l}{v} < t < \frac{(2N+3)l}{v}$$

Steady State Value and Error Term, Example

With algebra,

$$= \frac{Z_T}{Z_T + Z_S}(1 - \underbrace{(K_{rT} \cdot K_{rS})^{N+1}}) \qquad \text{for} \quad \frac{(2N+1)l}{v} < t < \frac{(2N+3)l}{v}$$

$\uparrow$ Attenuation in steady state

Error $\longrightarrow$ *0 as N* $\longrightarrow \infty$.

For $Z_S = 0,$ We get $K_{rS} = -1$

$$\text{Example}: Z_0 = 100,\ Z_T = 1900,\ Z_S = 0,\ \mathrm{K}_{\mathrm{rS}} = -1,\ K_{rT} = 0.9$$

$$V_l(t) = \frac{Z_T}{Z_T + Z_S}(1 - (-0.9)^{N+1})$$

$$\text{For error} < 10\% : N + 1 = \ln(0.1) / \ln(0.9) = 21.8,\ \ \mathrm{t} = 42.6(l/v)$$

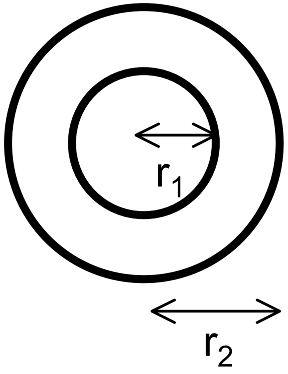
Example Continued

- For 2 foot line, this could be order of 85ns
- What to do?
  - » Wait a long time
  - » Real wire will have some resistance
  - » *CS solution*: use 1900 Ohm transmission Line
  - » Reduce resistance of receiver to match t-line
    - – Increased power
    - – May be a problem with driver resistance (.e.g 10 to 50 Ohms)
  - » Increase resistance of driver to match t-line
    - – 100 Ohms in series with driver gives 2.5% error after only l/v, or after 2ns for one foot line (this is the minimum possible)

Transmission Line Equations for Wire

Coax $C = 2\pi\varepsilon / \ln(r_2 / r_1) = \varepsilon_r \cdot (55.6\,pF / m) / \ln(r_2 / r_1)$

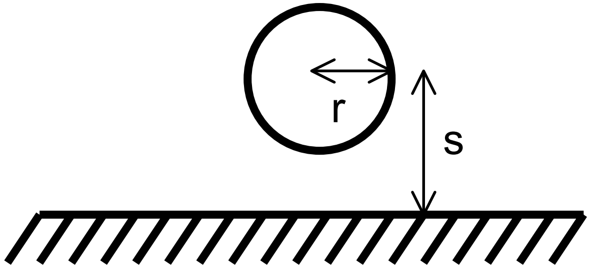
$$Z_0 = (60 / \sqrt{\varepsilon_r}) \cdot \ln(r_2 / r_1)$$



Wire over ground plane
(warning, not accurate for r close to s)

$$C / m = 2 \cdot \pi\varepsilon / \ln(2 \cdot s / r) = \varepsilon_r \cdot 55.6(pF / m) / \ln(2 \cdot s / r)$$

$$Z_0 = (60 / \sqrt{\varepsilon_r}) \cdot \ln(2 \cdot s / r)$$

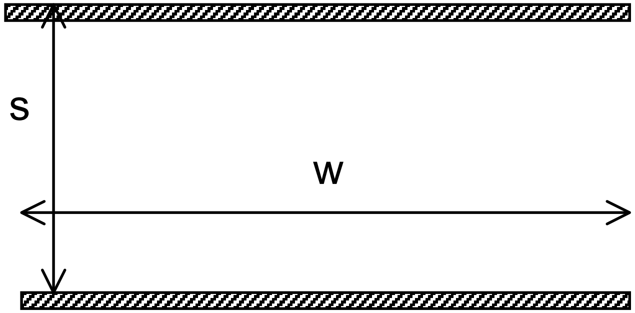


2 Wires: 1/2 capacitance of wire over ground plane

PC Board Equations

$$Z_0 \approx \frac{377}{\sqrt{\varepsilon_r}} \frac{s}{w}$$

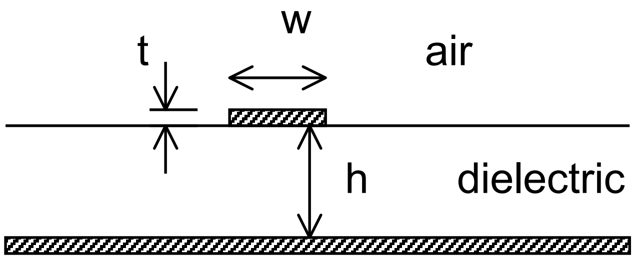
Parallel Plates
(ignoring fringing



PC Board Equations

(Careful here, approximations grossly wrong for some values: large w/h)

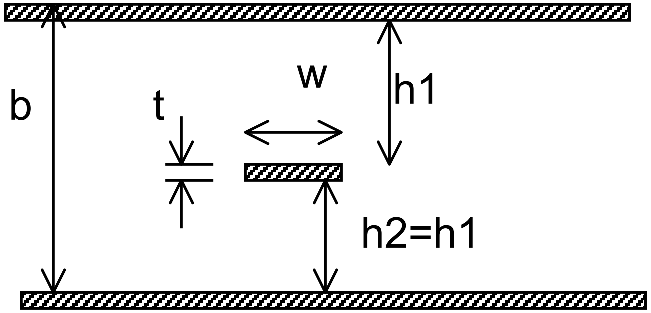
Microstrip



$$Z_0 \approx \frac{87}{\sqrt{\varepsilon_r + 1.41}} \ln\left(\frac{5.98 \cdot h}{0.8 \cdot w + t}\right)$$

$$t_{pd}(ns/ft) \approx 1.017\sqrt{0.475 \cdot \varepsilon_r + 0.67}$$

Stripline



$$Z_0 \approx \frac{60}{\sqrt{\varepsilon_r}} \ln\left(\frac{4 \cdot b}{0.67 \cdot \pi \cdot w(0.8 + t/w)}\right)$$

$$\text{for} \quad w/(b-t) < 0.35, \; t/b < 0.25$$

$$t_{pd}(ns/ft) \approx 1.017\sqrt{\varepsilon_r}$$