1. Classical entailment in PAC-Semantics. In this problem, we will relate classical entailment to entailment in PAC-Semantics.

(a) Let $A \subseteq X$ be any property. Suppose we know that $A$ is $(1 - \epsilon)$-valid with respect to $D$, i.e., $\Pr_{x \in D}[A(x) = 1] \geq 1 - \epsilon$ and that $A \models \varphi$ for some representation $\varphi(x)$. Prove that $\varphi$ is therefore also $(1 - \epsilon)$-valid with respect to $D$.

(b) Now suppose that for representations $\psi_1(x), \ldots, \psi_k(x)$, we know that each $\psi_i(x)$ is $(1 - \epsilon_i)$-valid with respect to a common distribution $D$ for some $\epsilon_i \in [0, 1]$, and for the knowledge base $KB = \{\psi_1, \ldots, \psi_k\}$, $KB \models \varphi$ for some representation $\varphi(x)$. Prove that then $\varphi(x)$ is $(1 - \epsilon')$-valid with respect to $D$ where $\epsilon' = \sum_{i=1}^{k} \epsilon_i$.

(c) Finally, suppose that for $KB = \{\psi_1, \ldots, \psi_k\}$ where $\psi_1, \ldots, \psi_k$ are consistent, $\sum_{i=1}^{k} \epsilon_i = \epsilon' < 1$ and for each $i$, $\psi_i$ is not entailed by $KB - \{\psi_i\}$. Show then that there exists a distribution $D$ on $X$ under which, for $\varphi = \psi_1 \land \cdots \land \psi_k$,

i. $\Pr_{x \in D}[\psi_i(x) = 1] = 1 - \epsilon_i$ for each $i = 1, \ldots, k$ and
ii. $\Pr_{x \in D}[\varphi(x) = 1] = 1 - \epsilon'$.

Thus: classical reasoning can be used freely in PAC-Semantics, and the “error” of the conclusions may depend additively on the error in the premises in general.

2. Clause Space = Tree Rank $+ 1$. The clause space of a treelike resolution refutation can be characterized cleanly in terms of the rank of the tree underlying the proof. Recall that the rank is defined inductively: leaves have rank 0, and any non-leaf node with subtrees of rank $r_1$ and $r_2$ has rank $\max\{r_1, r_2\}$ if $r_1 \neq r_2$, and otherwise has rank $r_1 + 1 (= r_2 + 1)$.

(a) Show that the graph of a treelike resolution refutation using clause space $s$ is a tree of rank $s - 1$.

(b) Show that if a treelike resolution refutation has a graph that is of rank $k$, then the treelike refutation can be carried out (in an appropriate order) using clause space $k + 1$. 