GENERAL

- PSET 1 Prob 6 Solutions posted for use in PSET 2.
- Typo in PSET 2 code: nted should return HxW array (not HxWx3)
- Still issues with Monday office hours location
  - Tentatively still at Jolley 431
  - Keep an eye out on Piazza
  - [If J431 is locked, check collaboration area outside J517]
- Recitation this Friday Oct 6.

HOMOGENEOUS CO-ORDINATES

- Useful way to think about 2-D Homogeneous Co-ordinates $\mathbb{P}^2$

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  x
  
  y
```

"Rays" in $\mathbb{R}^3$

- Cartesian form is "intersection" with plane $z = 1$.
- $(x, y, 0)$ are forms that are parallel to the $z = 1$ plane, intersect at infinity.
- 3-D Homogeneous Co-ordinates are rays in 4D, intersection with a hyper-plane.

QUICK WORD ABOUT NOTATION

- We assume vectors are column vectors.
- $p' = [x, y]$ implies a 2-D row vector (of size $1 \times 2$)
- $p = [x, y]^T = \begin{bmatrix} x \\ y \end{bmatrix}$ implies a 2-D column vector (of size $2 \times 1$)
HOMOGENEOUS CO-ORDINATES: 2D

Lines

Equation of a line in 2D:

\[ ax + by + c = 0 \]

Let \( p = [\alpha x, \alpha y, \alpha]^T \) be homogeneous co-ordinates of a point \((x, y)\). Then,

\[ l^T p = 0, \quad l = [a, b, c]^T \]

Interestingly, \( l \) is also defined "upto scale": \( l' = [\beta a, \beta b, \beta c]^T \) describes the same line as \( l \).

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HOMOGENEOUS CO-ORDINATES: 2D

Lines

Given two points \( p_1 \) and \( p_2 \), what is the homogeneous vector for the line joining them?

It has to be an \( l \) such that \( l^T p_1 = 0 \) and \( l^T p_2 = 0 \).

Is that sufficient to determine \( l \)?

Yes. Because, only need \( l \) upto scale.

Solution given by: \( l = p_1 \times p_2 \) (Vector Cross-product)

Recap: Writing \( u = [u_1, u_2, u_3]^T = u_1 \hat{i} + u_2 \hat{j} + u_3 \hat{k} \) and \( u = [v_1, v_2, v_3]^T = v_1 \hat{i} + v_2 \hat{j} + v_3 \hat{k} \)

\[
\begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
u_1 & u_2 & u_3 \\
v_1 & v_2 & v_3
\end{vmatrix} = (u_2v_3 - u_3v_2)\hat{i} + (u_3v_1 - u_1v_3)\hat{j} + (u_1v_2 - u_2v_1)\hat{k}
\]

\[
= [(u_2v_3 - u_3v_2), (u_3v_1 - u_1v_3), (u_1v_2 - u_2v_1)]^T
\]

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HOMOGENEOUS CO-ORDINATES: 2D

Lines

Given two lines \( l_1 \) and \( l_2 \), what is the homogeneous co-ordinate vector \( p \) for the point of their intersection?

Same idea: \( l_1^T p = p^T l_1 = 0 \) and \( p^T l_2 = 0 \)

\[
p = l_1 \times l_2
\]

- Cross product between two points gives us the line between them
- Cross product between two lines gives us the point common to both
- What happens if \( l_1 \) and \( l_2 \) are parallel?

Answer: Third co-ordinate of \( l_1 \times l_2 \) is 0. Point at infinity.
HOMOGENEOUS CO-ORDINATES: 2D

Transformations

- Translation:
  \[ x' = x - c_x, y' = y - c_y \]

\[ p' = \begin{bmatrix} 1 & 0 & -c_x \\ 0 & 1 & -c_y \\ 0 & 0 & 1 \end{bmatrix} p \]

- Verify this works for any scaled version of \( T \) above
- Verify this works for \( p = [ax, ay, a] \), for any \( a \neq 0 \)

HOMOGENEOUS CO-ORDINATES: 2D

Transformations

- Rotation Around the Origin
  \[ x' = x \cos \theta - y \sin \theta, y' = x \sin \theta + y \cos \theta \]

\[ p' = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} p \]

- Translation around a different point \( c_x, c_y \)?

\[ p' = \begin{bmatrix} 1 & 0 & c_x \\ 0 & 1 & c_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -c_x \\ 0 & 1 & -c_y \\ 0 & 0 & 1 \end{bmatrix} p \]

HOMOGENEOUS CO-ORDINATES: 2D

Transformations

- Euclidean Transformation

\[ p' = \begin{bmatrix} R & t \\ 0^T & 1 \end{bmatrix} p \]

- \( R \) is a \( 2 \times 2 \) rotation matrix, \( R^T R = I \)
- \( t \) is a \( 2 \times 1 \) translation vector
- \( 0^T \) here represents a \( 1 \times 2 \) row of two zeros
- Preserves orientation, lengths, areas

If \( R^T R = I \), \( R \) always of the form:

\[ R = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \]

HOMOGENEOUS CO-ORDINATES: 2D

Transformations

- Euclidean Transformation

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If \( R^T R = I \), \( R \) always of the form:

\[ R = \begin{bmatrix} -\cos \theta & -\sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \]
HOMOGENEOUS CO-ORDINATES: 2D

Transformations

- **Euclidean Transformation**
  \[ p' = \begin{bmatrix} R & t \\ 0^T & 1 \end{bmatrix} p \]
  - \(R\) is a \(2 \times 2\) rotation matrix, \(R^2 R = I\)
  - \(t\) is a \(2 \times 1\) translation vector
  - \(0^T\) here represents a \(1 \times 2\) row of two zeros
  - Preserves orientation, lengths, areas
- **Isometries**
  - \(R^2 R = I\) can also correspond to reflections
  \[ R = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \]
  - If we allow this in \(R\) above, more general than euclidean
  - Preserves lengths, areas, but not orientation.

What about scaling?

- Allow uniform scaling \(s\) along both co-ordinates:
  \[ p' = \begin{bmatrix} sR & t \\ 0^T & 1 \end{bmatrix} p \]
  Called a similarity: preserves ratio of lengths, angles.

HOMOGENEOUS CO-ORDINATES: 2D

Affine Transformation

\[ p' = \begin{bmatrix} A & t \\ 0^T & 1 \end{bmatrix} p \]

where \(A\) is a general invertible \(2 \times 2\).
- Preserves ratios of areas, parallel lines stay parallel.
- Prove that parallel lines stay parallel.
  - Consider the homogeneous vector \(q\) for intersection of two lines that are parallel.
  - The third co-ordinate of \(q\) is 0, because the lines don't intersect.
  - The affine transform doesn't change the third co-ordinate.
  - Hence, the lines still intersect at infinity after the transformation.

Most general form:

\[ p' = Hp \]

where \(H\) is a general invertible \(3 \times 3\) matrix.

- Called a projective transform or **homography**.
- All bets are off! Parallel lines can now intersect. Maps quadrilaterals to quadrilaterals.
- Defined upto scale. So 8 degrees of freedom.
- **Hierarchy of Transforms**
  - Translation (2 dof) < Euclidean (3 dof) < Affine (6 dof) < Homography (8 dof)
  - Defines mapping of co-ordinates of corresponding points in two images taken from different views:
    - If all corresponding points lie on a plane in the world.
    - If only the camera orientation has changed in two views (center is at the same place).
I know a bunch of pairs of points \((p'_i, p_i)\), and want to find \(H\) such that:

\[ p'_i \sim Hp_i, \quad \forall i \]

Equality only up to scale: how do you turn that into an equation?

- How many unknowns? 8 (defined up to scale)
- How many equations for four points? 8 (2 x 4)

But how do we write these equations for equality up to scale?

Recall: \[ u \times v = [(u_2v_3 - u_3v_2), (u_3v_1 - u_1v_3), (u_1v_2 - u_2v_1)]^T \]

This is a linear equation in the elements of \(H\).

Let \( h = [h_1, h_2, h_3, h_4, \ldots h_9] \) be a vector of the 9 elements of \(H\). Can write:

\[ A_j h = 0 \]

What is the size of \(A_j\)?

The cross product gives us 3 equations, so \(A_j\) is 3 \x 9.

But, one of the rows of \(A_j\) is a linear combination of the other (\(A_j\) has rank 2). Can choose to keep only two rows, or all three.

Stacking all the \(A_j\) matrices for all different correspondences, we get:

\[ Ah = 0 \]

\(A\) is \(2n \times 9\) or \(3n \times 9\) matrix, where \(n\) is number of correspondences. \(\text{Rank}(A)\) is at most \(2n\).

Rank exactly equal to \(2n\) if no three points are collinear.

So we have \(Ah = 0\) and want to find \(h\) up to scale. \(A\) has rank \(2n\) and \(h\) has 9 elements.

**Case 1:** \(n = 4\) non-collinear points.

- Trivial solution is \(h = 0\). But want to avoid this.
- Cast as finding \(Ah = 0\) such that \(\|h\| = 1\).
- Since \(A\) is exactly rank 8, there exists such a solution and it is unique (up to sign).
- Can find using eigen-decomposition / SVD.

\[ A = UDVT \text{ where } D \text{ is diagonal with last element 0. } h \text{ is the last column of } V. \]

**Case 2:** \(n > 4\) non-collinear points.

- Over-determined case. Want to find "best" solution.
- \(h = \arg \min_h \|Ah\|^2, \quad \|h\| = 1\)
- Same solution, except that instead of taking 0 singular value, we take minimum singular value.
- \(\|Ah\|^2 = (Ah)^T(Ah) = h^T(A^TA)h\)
- Minimized by unit vector corresponding to lowest eigenvalue of \(A^TA\), or lowest singular value of \(A\).
Estimation from Lines

- How does a homography transform a line:

\[ l^T p = 0 \iff l'^T p' = 0 \]

\[ l^T H^{-1} H p = 0 \implies (H^{-T} l)^T (H p) = 0 \]

\[ l' = H^{-T} l \implies l = H^T l' \]

- If we find four pairs of corresponding lines, we can get a similar set of equations for \( l_i = H^T l'_i \) as for points.
- Get equations from \( l_i \times (H^T l'_i) = 0 \) for elements of \( H \).

Other approaches:

- Instead of measuring \( \|AH\|^2 \), might want to measure explicit geometric distance.
- Minimize distance in mapped cartesian co-ordinates (re-projection error).
- Involves division, no longer linear in \( H \). Iterative methods.
- See “Multiple View Geometry in Computer Vision,” Hartley & Zisserman: Section 4.2 (or really, the whole book for a thorough discussion of geometry)

Next Time

- Estimate camera projection matrix
- Relate transformation between two views
- Automated matching to solve for depth: Stereo