Analyzing the Cost of Treap Operations

Recall last time that we saw how to implement UNION. We also saw that SPLIT and JOIN can be implemented in $O(\log |T|)$ time, where $T$ is the input argument. Today, we will discuss the cost of implementing UNION and INTERSECTION.

**UNION**($T_1, T_2$)
1. if $T_1$ is empty,
2. then return $T_2$
3. $(L_1, R_1, (k_1, v_1)) \leftarrow \text{EXPOSE}(T_1)$
4. $(L_2, v_2, R_2) \leftarrow \text{SPLIT}(T_2, k_1)$
5. $L \leftarrow \text{spawn \enspace UNION}(L_1, L_2)$
6. $R \leftarrow \text{UNION}(R_1, R_2)$
7. sync
8. return $\text{JOIN}(L, (k_1, v_1), R)$

Let’s recall the basic structure of UNION($T_1, T_2$).

- For $T_1$ with key $k_1$ and children $L_1$ and $R_1$ at the root, use $k_1$ to split $T_2$ into $L_2$ and $R_2$.
- Recursively find $L_u = \text{UNION}(L_1, L_2)$ and $R_u = \text{UNION}(R_1, R_2)$.
- Now $\text{JOIN}(L_u, k_1, R_u)$.

Pictorially, the process looks like this:
Note that each call to \textsc{Union} makes one call to \textsc{Split} and one to \textsc{Join} each which take $O(\log n)$ work, where $n$ is the size of $T_2$. We assume that $T_1$ is the smaller tree with size $m$. For starters, we’ll make the following assumptions to ease the analysis:

1. let’s assume that $T_1$ it is perfectly balanced, and
2. each time a key from $T_1$ splits $T_2$, it splits it exactly in half.

The work recurrence is then $W(m, n) = 2W(m/2, n/2) + \Theta(\log n)$. In addition, if $m = 1$, then $W(1, n) = O(\log(1 + n))$.

This recurrence deserves more explanation: When $m > 1$, expose gives us a perfect split, resulting in a key $k_1$ and two subtrees of size $n/2$ each, and by our assumption (which we shall soon eliminate), $k_1$ splits $T_2$ perfectly in half, so the subtrees that produces have size $m/2$. When $m = 1$, we know that expose give us two empty subtrees $L_1$ and $R_1$, which means that both \textsc{Union}($L_1, L_2$) and \textsc{Union($R_1, R_2$) will return immediately with values $L_2$ and $R_2$, respectively. Joining these together with $T_1$ costs at most $O(\log(1 + n))$ time.

Is this tree root or leaf dominated, or evenly sized? And how many levels will it have? It is not hard to see that this tree is dominated at the leaves. In fact, what’s happening is that when we get to the bottom level, each leaf in $T_1$ has to split a subtree of $T_2$ of size $n/m$. This takes $O(\log(1 + \frac{n}{m}))$ work. Since there are $O(m)$ such leaves, the total work at the leaves is $O(m \log(1 + \frac{n}{m}))$. Furthermore, the work going up the tree decreases geometrically.

To bound this more formally, since $T_1$ has $m$ keys and it is split exactly in half, it will have $\log_2 m$ levels, so if we start counting from level 0 at the root, we have that the $i$-th level has $2^i$ nodes. The bottom level will have $m$ nodes. Size of $T_2$ at each leaf is $n/2^{\log_2 m}$. The whole bottom level will cost

$$k_1 \cdot m \cdot \log \left(1 + \frac{n}{2^{\log_2 m}}\right) = k_1 \cdot m \cdot \log \left(1 + \frac{n}{m}\right).$$

Since the tree is dominated at the leaves, the total work will be $O(m \log(1 + \frac{n}{m}))$, as desired. Hence, if the trees satisfy our assumptions, we have that \textsc{Union} runs in $O(m \log(1 + \frac{n}{m}))$.

Of course, in reality, our keys in $T_1$ won’t split subtrees of $T_2$ in half every time. But it turns out this only helps. We won’t go through the rigorous analysis, but let us think informally. Instead of splitting exactly in half, $T_2$ splits unevenly. So now, let us say that the subtrees that reach the leaves
have sizes $n_1, n_2, \ldots, n_k$, where $\sum_{i=1}^{k} n_i \leq n$ since the subtrees are a partition of the original tree $T_2$. Therefore, the total cost of splitting these subtrees is

$$\sum_{i=1}^{k} k_1 \log(n_i) \leq \sum_{i=1}^{k} k_1 \log(n/k),$$

where we use the inequality of arithmetic and geometric means (or, the AM-GM inequality).\footnote{Technically, we are using the fact that the logarithm function is concave and applying a variant of the so-called “Jensen’s inequality.”} This shows that the total work is $O(m \log(1 + \frac{n}{m}))$.

Now, what is the work at the internal nodes? At each level of the tree, subtree sizes of $T_2$ sum up to $n$ and you can apply Jensen’s inequality again to get the same result to bound the work at each level. Thus, the tree is leaf dominated, even if $T_2$ is not split evenly.

What about $T_1$? In actuality, $T_1$ doesn’t have to be perfectly balanced as we assumed. For treaps, we know that the depth of $T_1$ tree is at most $c \log m$ for some constant $c$. Therefore, the depth of the above recursion tree will also be $c \log m$. Again, at every level, the total sum of the nodes from $T_1$ is also at most $n$. Therefore, the recursion tree is still leaf dominated, and we can use the same analysis as above.

What about the span? Well, at each level of recurrence, the span is $\Theta(\log n)$ with $c \log m$ levels, so the span is $O(\log^2 n)$. Actually, the span can be improved to $O(\log n)$ by changing the the algorithm slightly (akin to how one gets the $O(\log n)$ span for the MERGE routine in mergesort), but it’s more complicated, so we won’t get into it.

Note that both the work and span are high-probability bounds. The cost of $f(n)$ in the recurrence above is dominated by the cost of split, which is $c \log n$ with high probability. Thus, in order to get the bound on work and span, we need to argue that at each level of recurrence, the bound indicated by $f(n)$ holds with high probability. To get the probability of getting the final bounds on work and span, we take a union bound across all levels of recurrence.

In summary, this means that UNION can be implemented in $O(m \log(n/m))$ work and span $O(\log n)$. The same holds for the other similar operations (e.g. INTERSECTION).

### Max Flow

In this lecture, we discussed serial algorithm for computing max flow, which can be found in the notes for lecture 22.